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Weighted Hardy-type Inequalities with Sharp Constants in L_p Spaces

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Abstract

Error estimate and the rate of convergence are very important in the framework of numerical analysis. Without doubt having enough information of the most important inequalities play an important role in achieving better bound in numerical algorithms. This paper focuses on Hardy's inequality associated with the Jacobi weight function $\omega^{\alpha,\beta} = (1-x)^{\alpha}(1+x)^{\beta}$ with $I := (-1, 1)$.

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1. Introduction

Hardy-type inequalities have attracted a lot of interest during all the years from the dramatic prehistory (see until Hardy discovered his famous inequality in 1925 [1]) to a still very active research (see [2, 3]).

We now recall that classical one-dimensional Hardy inequality [4]: let $1 < p < \infty$, $f(t) \geq 0$, and $F(x) = \int_0^x f(t)dt$, then

$$\int_0^{\infty} \Big(\frac{F(x)}{x}\Big)^p dx \leq \Big(\frac{p}{p-1}\Big)^p \int_0^{\infty} F'(x)dx.$$

In this work, we give an extension of this inequalities for the weight functions $\omega(x) = (x-a)^{\alpha}$ and $\omega(x) = (b-x)^{\alpha}$.

2. Main result

Let Ω be a Lebesgue-measurable subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with non-empty interior, and let f be a Lebesgue measurable function on Ω.

Definition 2.1. For $1 \leq p \leq \infty$ and the positive weight function $\omega(x)$ (a.e.), let

$$L^p_{\omega}(\Omega) := \{f : f \textit{ is measurable on } \Omega \textit{ and } \|f\|_{p,\omega} < \infty\},$$

* speaker

where for $1 \leq p \leq \infty$,

$$\|f\|_{p,\omega} := \Big(\int_{\Omega} |f(x)|^p \omega(x)dx\Big)^{1/p},$$

and

$$\|f\|_{\infty} := ess\ sup_{x\in X}|f(x)|.$$

Definition 2.2. The Sobolev space $H^m_\omega(\Omega)$ with $m \in \mathbb{N}$ is the space of functions $f \in L^2_\omega(\Omega)$ such that all the distributional derivatives of order up to m can be represented by functions in $L^2_\omega(\Omega)$. that is,

$$H^m_\omega(\Omega) = \{f \in L^2_\omega(\Omega) :\ D^\gamma f \in L^2_\omega(\Omega)\ for\ 0 \leq |\gamma| \leq m\},$$

equipped with the norm

$$\|f\|_{m,\omega} = \Big(\sum_{|\gamma|=0}^{m} \|D^\gamma f\|_{2,\omega}\Big)^{1/2}.$$

Definition 2.3. The space $H^m_{0,\omega}$ is defined as the closure of $C^\infty_0(\Omega)$ in $H^m_\omega(\Omega)$, where $C^\infty_0(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω.

Theorem 2.4. *Suppose that $a < b$ be two real numbers, $1 < p < \infty$ and $q = \frac{p}{p-1}$. Let $\alpha < \frac{p}{q}$. Then for any $f \in L^p(a,b)$ with $\omega(x) = (x-a)^\alpha$, we have the following Hardy inequalities:*

$$\int_a^b \Big(\frac{1}{x-a}\int_a^x f(t)dt\Big)^p (x-a)^\alpha dx \leq \Big(\frac{1}{1-\frac{q}{p}\alpha}\Big)\Big(\frac{p}{p-1}\Big)^p \int_a^b f^p(x)(x-a)^\alpha dx. \quad (1)$$

Similarly, for any $f \in L^p(a,b)$ with $\omega(x) = (b-x)^\alpha$, we have

$$\int_a^b \Big(\frac{1}{b-x}\int_x^b f(t)dt\Big)^p (b-x)^\alpha dx \leq \Big(\frac{1}{1-\frac{q}{p}\alpha}\Big)\Big(\frac{p}{p-1}\Big)^p \int_a^b f^p(x)(b-x)^\alpha. \quad (2)$$

Proof. Firstly we prove the first inequality. The prove of the second inequality is similiar to the first one. Pick $0 < \beta < \frac{1}{q}$, we will specifie it later. Define $F(x) = (x-a)^{(\alpha/p)-1}\int_a^x f(t)dt$. We start using Hölder's inequality

$$\begin{aligned}|(x-a)^{1-(\alpha/p)}F(x)| = |\int_a^x f(t)dt| &\leq \int_a^x |f(t)(t-a)^\beta||(t-a)-\beta|dt \\ &\leq \Big(\int_a^x |f(t)|^p(t-a)^{p\beta}dt\Big)^{1/p}\Big(\int_a^x (t-a)^{-q\beta}\Big)^{1/q} \\ &\leq \Big(\int_a^x |f(t)|^p(t-a)^{p\beta}dt\Big)^{1/p}\Big(\frac{(x-a)^{1-q\beta}}{1-q\beta}\Big)^{1/q} \\ &= (1-q\beta)^{-1/q}\Big(\int_a^x |f(t)|^p(t-a)^{p\beta}dt\Big)^{1/p}(x-a)^{1/q-\beta},\end{aligned}$$

hence

$$|F(x)| \leq (1-q\beta)^{-1/q}\Big(\int_a^x |f(t)|^p (t-a)^{p\beta} dt\Big)^{1/p} (x-a)^{-\beta-((1-\alpha)/p)}.$$

Therefore

$$|F(x)|^p \leq (1-q\beta)^{-p/q}\Big(\int_a^x |f(t)|^p (t-a)^{p\beta} dt\Big)(x-a)^{-p\beta-(1-\alpha)}.$$

Integrating and using Fubini's theorem, we get

$$\begin{aligned}
\int_a^b |F(x)|^p dx &\leq (1-q\beta)^{-p/q} \int_a^b \Big(\int_a^x |f(t)|^p (t-a)^{p\beta}(x-a)^{-p\beta-(1-\alpha)} dt\Big) dx \\
&\leq (1-q\beta)^{-p/q} \int_a^b \Big(\int_t^b |f(t)|^p (t-a)^{p\beta}(x-a)^{-p\beta-(1-\alpha)} dx\Big) dt \\
&\leq (1-q\beta)^{-p/q} \int_a^b |f(t)|^p (t-a)^{p\beta}\Big(\int_t^b (x-a)^{-p\beta-(1-\alpha)} dx\Big) dt \\
&\leq (1-q\beta)^{-p/q} \int_a^b |f(t)|^p (t-a)^{p\beta}\Big(\frac{1}{(\alpha-p\beta)}((b-a)^{\alpha-p\beta}-(t-a)^{\alpha-p\beta})\Big) dt \\
&= \frac{(1-q\beta)^{-p/q}}{(\alpha-p\beta)}\Big(\int_a^b |f(t)|^p (t-a)^{p\beta}(b-a)^{\alpha-p\beta} dt - \int_a^b |f(t)|^p (t-a)^{p\beta}(t-a)^{\alpha-p\beta} dt\Big) \\
&= -\frac{(1-q\beta)^{-p/q}}{(\alpha-p\beta)}\Big(\int_a^b |f(t)|^p (t-a)^{\alpha} dt - \int_a^b |f(t)|^p (t-a)^{p\beta}(t-a)^{\alpha-p\beta} dt\Big) \\
&\leq -\frac{(1-q\beta)^{-p/q}}{(\alpha-p\beta)}\Big(\int_a^b |f(t)|^p (t-a)^{\alpha} dt\Big),
\end{aligned}$$

and finally

$$\int_a^b |F(x)|^p dx \leq \frac{(1-q\beta)^{-p/q}}{(p\beta-\alpha)}\Big(\int_a^b |f(t)|^p (t-a)^{\alpha} dt\Big).$$

Now for a sharper bound in above inequality, we pick $\beta := \frac{1}{pq} < \frac{1}{q}$ in numerator and $\beta := \frac{1}{q}$ in denominator, to get

$$(1-q\beta)^{-p/q}(p\beta)^{-1} = (1-\frac{1}{p})^{-1/q} q = q^{p/q} q = q^{1+p(1-1/p)} = q^p,$$

and

$$(1-(p\beta)^{-1}\alpha) = 1 - \frac{q}{p}\alpha.$$

This completes the proof. □

3. Application

In this section, we apply the above Hardy inequality to derive some useful inequalities associated with the Jacobi weight function $\omega^{\alpha,\beta} = (1-x)^{\alpha}(1+x)^{\beta}$ with $I := (-1, 1)$. The following inequalities can be found in [5].

Lemma 3.1. *If* $-1 < \alpha, \beta < 1$, *then*

$$\|f\|_{2,\omega^{\alpha-2,\beta-2}} \leq c\|f'\|_{2,\omega^{\alpha,\beta}}, \quad \forall f \in H^1_{0,\omega^{\alpha,\beta}}(I), \tag{3}$$

which implies the poincaré-type inequality:

$$\|f\|_{2,\omega^{\alpha,\beta}} \leq c\|f'\|_{2,\omega^{\alpha,\beta}}, \quad \forall f \in H^1_{0,\omega^{\alpha,\beta}}(I), \tag{4}$$

Proof. Taking $a = -1$, $b = 1$ and $\phi = \frac{df}{dx}$ in (2) yields that for $\alpha < 1$,

$$\int_0^1 f^2(x)(1-x)^{\alpha-2}dx \leq c \int_0^1 f'^2(x)(1-x)^{\alpha}dx.$$

hence,

$$\begin{aligned}\int_0^1 f^2(x)(1-x)^{\alpha-2}(1+x)^{\beta-2}dx &\leq c\int_0^1 f^2(x)(1-x)^{\alpha-2}dx \\ &\leq c\int_0^1 f'^2(x)(1-x)^{\alpha}dx \\ &\leq c\int_0^1 f'^2(x)(1-x)^{\alpha}(1+x)^{\beta}dx.\end{aligned}$$

Similarly, for $\beta < 1$, we use (2) to derive

$$\int_{-1}^0 f^2(x)(1-x)^{\alpha-2}(1+x)^{\beta-2}dx \leq c\int_{-1}^0 f'^2(x)(1-x)^{\alpha}(1+x)^{\beta}dx.$$

A combination of the above two inequalities leads to (3).

In view of $\omega^{\alpha,\beta}(x) < \omega^{\alpha-2,\beta-2}(x)$, (4) follows from (3). □

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