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Using a refinable function for the construction of multiresolution analysis in $L^2(G)$

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Abstract

For any locally compact abelian (LCA) and second countable group G, we aim to construct a multiresolution analysis (MRA) in $L^2(G)$ by Riesz family of shifts of a refinable function $\varphi \in L^2_\circ(G)$ based on a uniform lattice L in G that at first, we investigate certain Banach spaces $L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$.

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1. Introduction

An MRA refers to the family $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L^p(G)$, $1 \leqslant p \leqslant \infty$, which is generated by the lattice translates of the dilates of a function φ. In such conditions, there is a function φ in V_0 that lattice translates of φ form an unconditional basis for V_0. Such a function is called scaling function. The idea of MRA was introduced by Meyer and Mallat, which provides a natural framework for construction of wavelet bases. ZJia and Micchelli in [1] proved that the Riesz family of integer translates of a certain basis refinable function are sufficient to lead to a multiresolution analysis of $L^p(\mathbb{R}^s)$ for $1 \leqslant p < \infty$. Later Zhou[4] developed this theory to the case $p = \infty$. In 1994 Dahlke generalized the definition of MRA to LCA groups, and he displayed that under specified conditions, the generalized B-splines generated an MRA. Kamyabi Gol and Raisi Tousi illustrated in [2] the conditions under which a function generates an MRA based on the spectral functions in the case of LCA groups.
In this paper, compared to [2] under a weaker assumption (Riesz family vs. orthonormality), but an additional assumption (refinability of φ), we discuss the construction of a multiresolution approximation in $L^2(G)$, by Riesz family of shifts of a certain refinable function φ.

2. Preliminaries and related background

Let G be an LCA group with the identity 1_G and the dual group $\hat{G}$. For a closed subgroup H of G, let $H^\perp$:=$\{\xi \in \hat{G};\ \xi(H) = \{1\}\}$, denotes as the the annihilator of

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H in $\hat{G}$. A discrete subgroup L of G is called a uniform lattice if it is co-compact. Now a fundamental domain for a uniform lattice L in G, is a measurable set S_L in G, such that every $x \in G$ can be uniquely written as $x = ks$, for $k \in L$ and $s \in S_L$.Consider the dilation operator $D : L^p(G) \longrightarrow L^p(G)$ by $Df(x) = \delta_\alpha^{\frac{1}{p}} f(\alpha(x))$, $1 \leqslant p < \infty$, (in fact, δ_α is a proper positive constant depending on α such that the operator D becomes an isometrically isomorphism). Now, we introduce the notion of multiresolution approximation in $L^2(G)$, following [3]. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ forms a multiresolution approximation of $L^2(G)$ if it satisfies the following conditions:

(i) $V_j \subseteq V_{j+1}, \forall j \in \mathbb{Z}$.
(ii) $f \in V_j \Longrightarrow D^j T_k D^{j^{-1}} f \in V_j$, for all $j \in \mathbb{Z}, k \in L$.
(iii) $f \in V_j \Longleftrightarrow \delta_\alpha^{-\frac{1}{2}} Df \in V_{j+1}$.
(iv) There is an isomorphism from $l^2(L)$ onto V_0 which commutes with shift operators.
(v) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$.
(vi) $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(G)$,

We recall that for a locally compact group G, a topological automorphism $\alpha : G \to G$ is said to be contractive if $\lim_{n\to\infty} \alpha^n(x) = 1_G$ for all $x \in G$.
Now, we introduce Banach spaces $L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$. For a function φ on G and uniform lattice L in G, let

$$\varphi^\circ(x) := \sum_{k\in L} |\varphi(k^{-1}x)|,$$

then φ° is a L-periodic function. Write

$$|\varphi|_p := \|\varphi^\circ\|_{L^p(S_L)},$$

and let,

$$L^p_\circ(G) = \{\varphi : G \longrightarrow \mathbb{C}; \quad |\varphi|_p < \infty\} \quad (1 \leqslant p \leqslant \infty).$$

$L^p_\circ(G)$ equipped with the norm $|.|_p$, is a Banach space, and obviously $\|\varphi\|_p \leqslant |\varphi|_p$, for all $1 \leqslant p \leqslant \infty$.
Note that $L^1_\circ(G) = L^1(G)$. Also, if $\varphi \in L^p(G)$ is compactly supported, then $\varphi \in L^p_\circ(G)$, for all $1 \leqslant p \leqslant \infty$.
Now, semidiscrete convolution $\varphi *' a$ is defined by $\sum_{k\in L} \varphi(k^{-1}.)a(k)$ for all $\varphi \in L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$, and a sequence $a \in l^\infty(L)$. We also denote by $\varphi *'$ the mapping $a \to \varphi *' a$, $a \in l^\infty(L)$.
We recall that the shifts of φ, under the lattice L in G is said to be a Riesz family of $L^p(G)$, if there exist constants $A_p, B_p > 0$ such that

$$A_p\|a\|_p \leqslant \|\varphi *' a\|_p \leqslant B_p\|a\|_p \quad (1 \leqslant p \leqslant \infty),$$

for all $a \in l^p(L)$.
Let $S_p(\varphi)$ be the image of $l^p(L)$ of the mapping $\varphi *'$. In this case the set of shifts of φ under the lattice L in G is a Riesz basis of $S_p(G)$.

3. Multiresolution Analysis

In this section for a refinable function $\varphi \in L^2_\circ(G)$, we consider $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, where D is dilation operator. We construct an MRA of $L^2(G)$ by a Riesz family of shifts of φ under the lattice L in G.
A function $\varphi \in L^p_\circ(G)$ is said to be refinable, if it satisfies the following refinement equation:

$$\begin{aligned}\varphi &= \sum_{k\in L} b(k) D T_k \varphi(.)\\ &= \sum_{k\in L} \delta_\alpha^{\frac{1}{p}} b(k)\varphi(k^{-1}\alpha(.)),\end{aligned} \tag{1}$$

for some $b \in l^1(L)$, that is called the mask of the refinement equation.

Theorem 3.1. *Let $\varphi \in L^2(G)$, $V_0 = S_2(G)$ and $V_j = D^j V_0$. If φ is refinable and shifts of φ are Riesz family under the lattice L in G, then $(V_j)_{j\in\mathbb{Z}}$ forms a multiresolution approximation of $L^2(G)$.*

Theorem 3.2. *Let $\varphi \in L^2_\circ(G)$, $V_0 = S_2(G)$ and $V_j = D^j V_0$. If the set of shifts of φ is a Riesz family, then $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$*

Remark 3.3. *Theorem 3.2 is valid for every function $\varphi \in L^p(G)$, $1 \leqslant p < \infty$. But in the case $p = \infty$, Theorem 3.2 may fail to hold. For example let φ be the characteristic function of interval $[0,1) \subset \mathbb{R}$ and let $V_0 = S_\infty(\varphi)$, $V_j = D^j V_0$. The set of integer translates of φ is a Riesz basis of V_0, but $1 \in V_j$ for all $j \in \mathbb{Z}$.*

To prove property (vi) we need the following propositions. The following proposition shows that for a refinable function $\varphi \in L^1(G)$, $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp \setminus \{1_{\hat{G}}\}$.

Proposition 3.4. *If $\varphi \in L^1(G)$ is refinable and $\alpha : G \to G$ is a topological automorphism such that $\hat{\alpha}^{-1}$, is contractive and $\hat{\alpha}(L^\perp) \subseteq L^\perp$, then $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^\perp \setminus \{1_{\hat{G}}\}$. Moreover,*

$$\sum_{k\in L} \varphi(k^{-1}.) = \hat{\varphi}(1_{\hat{G}}).$$

Proposition 3.5. *Let $\varphi \in L^p_\circ(G)$, $1 \leqslant p \leqslant \infty$, and the shifts of φ be a Riesz family of $L^p(G)$ under the lattice L in G; then, for all $\xi \in \widehat{G}$,* $\sup_{\eta\in L^\perp} |\hat{\varphi}(\xi\eta)| > 0,$

Propositions 3.4 and 3.5 guarantee $\hat{\varphi}(1_{\hat{G}}) \neq 0$. After normalization, we may assume $\hat{\varphi}(1_{\hat{G}}) = 1$; thus, we can state property (vi) as follows:

Theorem 3.6. *If $\varphi \in L^2_\circ(G)$ is refinable, such that shifts of φ under the lattice L are Riesz family. Then $\bigcup_{j\in\mathbb{Z}} V_j$, is dense in $L^2(G)$.*

Example 3.7. *Let G be the following LCA group,*

$$G = \{x = (x_n)_{n\in\mathbb{Z}},\ x_n \in Z_2 = \{0,1\},\ \exists N \in \mathbb{Z}\ \ s.t.\ \forall n > N \Rightarrow x_n = 0\},$$

with the operation given by

$$(x^1 + x^2)_n = x^1_n + x^2_n \quad mod\ 2.$$

We identify G with $[0, \infty)$ *as a measure space by* $x \to |x|$ *where* $|x| = \sum_{j\in\mathbb{Z}} x_j 2^j$. *This induces the Haar measure of* $[0, \infty)$ *on G. We will be interested in the following subgroups,*

$$L = \{x \in G, x_j = 0\ for\ j < 0\},$$
$$D = \frac{G}{L} = \{x \in G, x_j = 0\ for\ j \geqslant 0\}.$$

The subgroup D is known as the Cantor group. We have that L is countable, closed, and discrete, and that D is compact. Consider the Hilbert space $H = L^2(G, \mu_G)$. *The dilation* $\rho : H \to H$ *and translation* $T : H \to H$ *are defined respectively by* $(\rho f)(x))_j = f(x_{j-1})$ *and* $T_k f(x) = f(x - k)$ *for* $f \in H, x \in G, k \in L$. *Let the scaling function be* $\phi(x) = \chi_D(x)$, *the characteristic function of D. We have* $(\rho^{-1}\varphi)(x) = \varphi(x) + \varphi(x + 1)$, *so* χ_D *is satisfied in refinable equation and shifts of* φ *are an orthonormal basis of H. suppose* $V_0 = S_2(\varphi)$ *and* $V_j = D^j V_0$, *therefore by Theorem 3.1,* V_j*s in which* $j \in \mathbb{Z}$, *construct a multiresolution approximation of H.*

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