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Using a refinable function for the construction of multiresolution analysis in $L^2(G)$

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Abstract

For any locally compact abelian (LCA) and second countable group G, we aim to construct a multiresolution analysis (MRA) in $L^2(G)$ by Riesz family of shifts of a refinable function $\varphi \in L^2_{\circ}(G)$ based on a uniform lattice *L* in *G* that at first, we investigate certain Banach spaces $L^p_{\circ}(G)$, $1 \le p \le \infty$.

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1. Introduction

An MRA refers to the family $\{V_j\}_{j\in\mathbb{Z}}$ of subspaces of $L^p(G)$, $1 \le p \le \infty$, which is generated by the lattice translates of the dilates of a function φ . In such conditions, there is a function φ in V_0 that lattice translates of φ form an unconditional basis for V_0 . Such a function is called scaling function. The idea of MRA was introduced by Meyer and Mallat, which provides a natural framework for construction of wavelet bases. ZJia and Micchelli in [1] proved that the Riesz family of integer translates of a certain basis refinable function are sufficient to lead to a multiresolution analysis of $L^p(\mathbb{R}^s)$ for $1 \le p < \infty$. Later Zhou[4] developed this theory to the case $p = \infty$. In 1994 Dahlke generalized the definition of MRA to LCA groups, and he displayed that under specified conditions, the generalized B-splines generated an MRA. Kamyabi Gol and Raisi Tousi illustrated in [2] the conditions under which a function generates an MRA based on the spectral functions in the case of LCA groups.

In this paper, compared to [2] under a weaker assumption (Riesz family vs. orthonormality), but an additional assumption (refinability of φ), we discuss the construction of a multiresolution approximation in $L^2(G)$, by Riesz family of shifts of a certain refinable function φ .

2. Preliminaries and related background

Let G be an LCA group with the identity 1_G and the dual group \hat{G} . For a closed subgroup H of G, let $H^{\perp}:=\{\xi \in \hat{G}; \xi(H) = \{1\}\}$, denotes as the the annihilator of

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H in \hat{G} . A discrete subgroup *L* of *G* is called a uniform lattice if it is co-compact. Now a fundamental domain for a uniform lattice *L* in *G*, is a measurable set *S_L* in *G*, such that every $x \in G$ can be uniquely written as x = ks, for $k \in L$ and $s \in S_L$.Consider the dilation operator $D : L^p(G) \longrightarrow L^p(G)$ by $Df(x) = \delta_\alpha^{\frac{1}{p}} f(\alpha(x))$, $1 \leq p < \infty$, (in fact, δ_α is a proper positive constant depending on α such that the operator *D* becomes an isometrically isomorphism). Now, we introduce the notion of multiresolution approximation in $L^2(G)$, following [3]. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(G)$ forms a multiresolution approximation of $L^2(G)$ if it satisfies the following conditions:

- (*i*) $V_j \subseteq V_{j+1}, \forall j \in \mathbb{Z}.$
- (*ii*) $f \in V_j \Longrightarrow D^j T_k D^{j^{-1}} f \in V_j$, for all $j \in \mathbb{Z}, k \in L$.
- (*iii*) $f \in V_j \iff \delta_{\alpha}^{-\frac{1}{2}} Df \in V_{j+1}$.
- (*iv*) There is an isomorphism from $l^2(L)$ onto V_0 which commutes with shift operators.
- $(v) \quad \bigcap_{j \in \mathbb{Z}} V_j = \{0\}.$
- (vi) $\overline{\bigcup_{j\in\mathbb{Z}}V_j} = L^2(G),$

We recall that for a locally compact group *G*, a topological automorphism $\alpha : G \to G$ is said to be contractive if $\lim_{n\to\infty} \alpha^n(x) = 1_G$ for all $x \in G$.

Now, we introduce Banach spaces $L^p_{\circ}(G)$, $1 \leq p \leq \infty$. For a function φ on G and uniform lattice L in G, let

$$\varphi^{\circ}(x) := \sum_{k \in L} |\varphi(k^{-1}x)|,$$

then φ° is a *L*-periodic function. Write

$$\varphi|_p := \|\varphi^\circ\|_{L^p(S_L)},$$

and let,

$$L^p_\circ(G) = \{ \varphi : G \longrightarrow \mathbb{C}; \quad |\varphi|_p < \infty \} \quad (1 \le p \le \infty).$$

 $L^p_{\circ}(G)$ equipped with the norm $|.|_p$, is a Banach space, and obviously $||\varphi||_p \leq |\varphi|_p$, for all $1 \leq p \leq \infty$.

Note that $L^1_{\circ}(G) = L^1(G)$. Also, if $\varphi \in L^p(G)$ is compactly supported, then $\varphi \in L^p_{\circ}(G)$, for all $1 \leq p \leq \infty$.

Now, semidiscrete convolution $\varphi *' a$ is defined by $\sum_{k \in L} \varphi(k^{-1}) a(k)$ for all $\varphi \in L^p_{\circ}(G)$, $1 \leq p \leq \infty$, and a sequence $a \in l^{\infty}(L)$. We also denote by $\varphi *'$ the mapping $a \to \varphi *' a$, $a \in l^{\infty}(L)$.

We recall that the shifts of φ , under the lattice *L* in *G* is said to be a Riesz family of $L^p(G)$, if there exist constants $A_p, B_p > 0$ such that

$$A_p ||a||_p \leq ||\varphi *' a||_p \leq B_p ||a||_p \quad (1 \leq p \leq \infty),$$

for all $a \in l^p(L)$.

Let $S_p(\varphi)$ be the image of $l^p(L)$ of the mapping $\varphi *'$. In this case the set of shifts of φ under the lattice L in G is a Riesz basis of $S_p(G)$.

Multiresolution Analysis

3. Multiresolution Analysis

In this section for a refinable function $\varphi \in L^2_{\circ}(G)$, we consider $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, where D is dilation operator. We construct an MRA of $L^2(G)$ by a Riesz family of shifts of φ under the lattice L in G.

A function $\varphi \in L^p_{\circ}(G)$ is said to be refinable, if it satisfies the following refinement equation:

$$\varphi = \sum_{k \in L} b(k) DT_k \varphi(.)$$
$$= \sum_{k \in L} \delta_{\alpha}^{\frac{1}{p}} b(k) \varphi(k^{-1} \alpha(.)), \qquad (1)$$

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for some $b \in l^1(L)$, that is called the mask of the refinement equation.

Theorem 3.1. Let $\varphi \in L^2(G)$, $V_0 = S_2(G)$ and $V_j = D^j V_0$. If φ is refinable and shifts of φ are Riesz family under the lattice L in G, then $(V_j)_{j \in \mathbb{Z}}$ forms a multiresolution approximation of $L^2(G)$.

Theorem 3.2. Let $\varphi \in L^2_{\circ}(G)$, $V_0 = S_2(G)$ and $V_j = D^j V_0$. If the set of shifts of φ is a Riesz family, then $\bigcap_{i \in \mathbb{Z}} V_i = \{0\}$

Remark 3.3. Theorem 3.2 is valid for every function $\varphi \in L^p(G)$, $1 \leq p < \infty$. But in the case $p = \infty$, Theorem 3.2 may fail to hold. For example let φ be the characteristic function of interval $[0, 1) \subset \mathbb{R}$ and let $V_0 = S_{\infty}(\varphi)$, $V_j = D^j V_0$. The set of integer translates of φ is a Riesz basis of V_0 , but $1 \in V_j$ for all $j \in \mathbb{Z}$.

To prove property (*vi*) we need the following propositions. The following proposition shows that for a refinable function $\varphi \in L^1(G)$, $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^{\perp} \setminus \{1_{\hat{G}}\}$.

Proposition 3.4. If $\varphi \in L^1(G)$ is refinable and $\alpha : G \to G$ is a topological automorphism such that $\hat{\alpha}^{-1}$, is contractive and $\hat{\alpha}(L^{\perp}) \subseteq L^{\perp}$, then $\hat{\varphi}(\eta) = 0$ for all $\eta \in L^{\perp} \setminus \{1_{\hat{G}}\}$. Moreover,

$$\sum_{k\in L}\varphi(k^{-1}.)=\hat{\varphi}(1_{\hat{G}}).$$

Proposition 3.5. Let $\varphi \in L^p_{\circ}(G)$, $1 \leq p \leq \infty$, and the shifts of φ be a Riesz family of $L^p(G)$ under the lattice L in G; then, for all $\xi \in \widehat{G}$, $\sup_{n \in L^{\perp}} |\hat{\varphi}(\xi \eta)| > 0$,

Propositions 3.4 and 3.5 guarantee $\hat{\varphi}(1_{\hat{G}}) \neq 0$. After normalization, we may assume $\hat{\varphi}(1_{\hat{G}}) = 1$; thus, we can state property (*vi*) as follows:

Theorem 3.6. If $\varphi \in L^2_{\circ}(G)$ is refinable, such that shifts of φ under the lattice L are Riesz family. Then $\bigcup_{j \in \mathbb{Z}} V_j$, is dense in $L^2(G)$.

Example 3.7. Let G be the following LCA group,

$$G = \{x = (x_n)_{n \in \mathbb{Z}}, x_n \in \mathbb{Z}_2 = \{0, 1\}, \exists N \in \mathbb{Z} \ s.t. \ \forall n > N \Rightarrow x_n = 0\},\$$

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with the operation given by

$$(x^1 + x^2)_n = x_n^1 + x_n^2 \mod 2.$$

We identify G with $[0, \infty)$ as a measure space by $x \to |x|$ where $|x| = \sum_{j \in \mathbb{Z}} x_j 2^j$. This induces the Haar measure of $[0, \infty)$ on G. We will be interested in the following subgroups,

$$L = \{x \in G, x_j = 0 \text{ for } j < 0\},\$$
$$D = \frac{G}{L} = \{x \in G, x_j = 0 \text{ for } j \ge 0\}$$

The subgroup D is known as the Cantor group. We have that L is countable, closed, and discrete, and that D is compact. Consider the Hilbert space $H = L^2(G, \mu_G)$. The dilation $\rho : H \to H$ and translation $T : H \to H$ are defined respectively by $(\rho f)(x)_j = f(x_{j-1})$ and $T_k f(x) = f(x - k)$ for $f \in H, x \in G, k \in L$. Let the scaling function be $\phi(x) = \chi_D(x)$, the characteristic function of D. We have $(\rho^{-1}\varphi)(x) = \varphi(x) + \varphi(x + 1)$, so χ_D is satisfied in refinable equation and shifts of φ are an orthonormal basis of H. suppose $V_0 = S_2(\varphi)$ and $V_j = D^j V_0$, therefore by Theorem 3.1, V_j s in which $j \in \mathbb{Z}$, construct a multiresolution approximation of H.

References

- R. Q. JIA, C. A. MICCHELLI, Using the refinement equation for the construction of prewavelets II: Powers of two, in Curves, Surfaces, P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, eds., Academic Press, New York, 1991, 209-246.
- [2] R. A. KAMYABI GOL, R. RAISI TOUSI, Some equivalent multiresolution conditions on locally compact abelian groups, Proc. Indian Acad. Sci. (Math. Sci.), 120(3)(2010), 317-331.
- [3] S. G. MALLAT, Multiresolution approximations and wavelet orthonormal bases of $L_2(\mathbb{R})$, Trans. Amer. Math. Soc., **315**(1989), 69-87.
- [4] D. X. ZHOU, Stability of refinable functions, multiresolution analysis and Haar bases, SIAM J. Math. Anal., 27(1996), 891-904.

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