Numerical Integration Using Daubechies' Wavelets by Linear Least square method

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Abstract
In this paper, we use a method based on Daubechies wavelets for obtaining numerical solution of definite integral. This approximation depends on pure scaling functions expansion. The Method transforms definite integrals to a system of algebraic equations. We apply least square method for solving linear system. In order to illustrate the efficiency and accuracy of the method a few test examples are given.

Keywords: Numerical integration, Daubechies' wavelets.

1.Introduction

Integration of a function on bounded interval is an important operation for many applied science problems such as physical problems. There are several numerical approximating method for numerical integration of a function. Most procedure for approximating value of definite integral are based on use the polynomial that approximate the function. Daubechies' wavelets are bases function that satisfy certain mathematical requirements and are used for approximation of a function and also the Daubechies wavelets bases are orthonormal bases that having compact support on [0,D-1], where the parameter D is called the Daubechies number or wavelet genus. The basic scaling function represented by $\varphi(x)$ and basic wavelet function represented by $\psi(x)$ such that they both satisfy the following two-scaling relations respectively

$$\varphi(x)=\sum_{l=0}^{D-1}a_l\varphi(2x-l), \tag{1}$$

$$\psi(x)=\sum_{l=0}^{D-1} b_l \psi(2x-l), \qquad (2)$$

Where the coefficients $\{a\}_{l=0}^{D-1}$ are called filter coefficients, and $b_l=(-1)^l a_{D-1-l}$ [1]. And the Daubechies orthonormal basis is formed by [2]

$$\varphi_{ij}(x)=2^{j/2}\varphi\left(2^j x-l\right), \qquad (3)$$

and

$$\psi_{ij}(x)=2^{j/2}\psi\left(2^j x-l\right). \qquad (4)$$

Denoted the set of orthogonal functions by $\{\varphi_{ij}(x), l\in \mathrm{Z}\}$ for a particular j, generates a space $V_j \subset L^2(\mathrm{R})$ and $W_j \subset L^2(\mathrm{R})$ is orthogonal complementary in V_{j+1}. Let P_j denote the orthogonal projection $L^2(\mathrm{R})\to V_j$. Then the vector space V_j have the following conditions defining a multiresolution analysis [2,3]:

(i) $V_j \subset L^2(\mathrm{R})$ and $\cdots \subset V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset\cdots$

(ii) $f(x)-P_j f(x)=\min f(x)-g(x)$, where $g(x)\in V_j$

(iii) $f(x)\in V_j \Leftrightarrow f(2x)\in V_{j+1}$ for all $j\in \mathrm{Z}$.

(iv) The projection $P_j f(x)$ converges to $f(x)$ when j tends to infinity:

$\lim_{j\to\infty} P_j f(x)=f(x)$ or $\cup_{j=0}^{\infty} V_j$ is dense in $L^2(\mathrm{R})$

such that

$$P_j f(x)=\sum_{l=-\infty}^{\infty} C_{j-1,l}\varphi_{j-1,l}(x)+\sum_{l=-\infty}^{\infty} d_{j-1,l}\psi_{j-1,l}(x), \qquad (5)$$

where the coefficients $c_{j,l}$ and $d_{j,l}$ are as follows:

$$c_{j,l}=\int_{-\infty}^{+\infty} f(x)\varphi_{j,l}(x)dx, \qquad (6)$$

and

$$d_{j,l}=\int_{-\infty}^{+\infty} f(x)\psi_{j,l}(x)dx. \qquad (7)$$

2.Method

Let φ be the basic scaling function of Daubechies number *D* and assumes that φ is known at the dyadic rationals $m/2^q$, $m=0,1,\ldots,(D-1)2^q$, for some chosen $q\in\mathrm{N}$. Pure scaling function expansions of a function *f(x)* can be written as follows [4]:

$$f(x)=\sum_{l=-\infty}^{+\infty}c_{j,l}\varphi_{j,l}(x),\ \ x\in\mathrm{R}, \tag{8}$$

at th e grid pint

$$x=x_k={k}/{2^r},k\in\mathrm{Z}, \tag{9}$$

where $r\in\mathrm{N}$ corresponds to some chosen resolution of the real line and

$$c_{j,l}=\int_{-\infty}^{+\infty}f(x)\varphi_{j,l}(x)dx\ . \tag{10}$$

when $x\in[a,b]$ for $a,b\in\mathrm{N}$, we impose the resolution 2^r on interval $[a,b]$, i.e.

$$x_k=k\frac{b-a}{2^r},k=0,1,\ldots,2^r-1. \tag{11}$$

so (8) takes the following form [5]:

$$f(x)=\sum_{l=2^j a-D+2}^{2^j b-1}c_{j,l}\varphi_{j,l}(x),x\in[a,b]. \tag{12}$$

So *f(x)* at grid point $x_k=k\frac{b-a}{2^r},k=0,1,\ldots,2^r-1$ and $r>j$ can be written as follows:

$$f\left(k\frac{b-a}{2^r}\right)=\sum_{l=2^j a-D+2}^{2^j b-1}c_{j,l}\varphi_{j,l}\left(k\frac{b-a}{2^r}\right), \tag{13}$$

if we use (3), so (13) takes the following form:

$$f\left(k\frac{b-a}{2^r}\right)=2^{j/2}\sum_{l=2^j a-D+2}^{2^j b-1}c_{j,l}\varphi\left(2^j k\frac{b-a}{2^r}-l\right), \tag{14}$$

which can be written in matrix form as follows:

$$f_r=2^{j/2}AC \tag{15}$$

where

$$f_r=\left(f(0),f\left(\frac{b-a}{2^r}\right),f\left(\frac{2(b-a)}{2^r}\right),\ldots,f\left(\frac{(2^r-1)(b-a)}{2^r}\right)\right)^T, \tag{16}$$

$$A=\begin{pmatrix} \varphi(-2^j a+D-2) & \cdots & \varphi(-2^j b+1) \\ \vdots & \ddots & \vdots \\ \varphi(\frac{(2^{j+1}-1)b-a(2^{j+1}+1)+2D-4}{2}) & \cdots & \varphi(\frac{-(b-a)}{2}) \end{pmatrix}_{2^r\times(2^j(b-a)+D-2)}, \quad (17)$$

$$c=\left(c_{j,2^j-D+2},c_{j,2^j-D+3},\dots,c_{j,2^j b-1}\right)^T. \quad (18)$$

Note that when we overhaul the system shown in (15) we find that the matrix A is not a square matrix. Therefore we can't solve the system shown in (15) , then we will use linear least square method to solve above system to obtain column-vector C. Linear least squares method is a procedure to determine the best fit line to data. The linear least square problem is formally defined as follows:

Given a real $m\times n$ matrix A and a real vector b, find a real n-vector x such that the function:

$\|r(x)\|=\|Ax-b\|$,

is minimized[6].

At end of this section we denoting the integrals of $\varphi(x)$ by $r(x)$ which written as [7]:

$$r(x)=\int_0^x \varphi(y)dy. \quad (19)$$

Let $f(x)\in V_j, x\in[a,b]$ for $a,b\in Z$, we can expand this function by (12). Integration both side of Eq. (12) from $x=a$ to $x=b$ and using integration theorem yields

$$\begin{aligned}\int_a^b f(x)dx &= \int_a^b \sum_{l=2^j a-D+2}^{2^j b-1} c_{j,l}\varphi_{j,l}(x)dx \\ &= 2^{j/2}\sum_{l=2^j a-D+2}^{2^j b-1} c_{j,l}\int_a^b \varphi(\underbrace{2^j x-l}_{u})dx \\ &= 2^{j/2}\sum_{l=2^j a-D+2}^{2^j b-1} c_{j,l}\,2^{-j}\int_a^b \varphi(u)du . \quad (20)\end{aligned}$$

Using definition of $r(x)$ shown in (19), Eq. (20) can be written as follows:

$$\int_a^b f(x)dx = 2^{-j/2}\sum_{l=2^j a-D+2}^{2^j b-1} c_{j,l}\left[r(2^j b-l)-r(2^j a-l)\right]. \quad (21)$$

which can be written in matrix form as follows:

$$\int_a^b f(x)dx = 2^{-j/2}RC . \quad (22)$$

where matrix C is the same shown in (15) and R is as follows:

$$R=\begin{pmatrix} r\left(2^j(b-a)+D-2\right)-r(D-2) \\ r\left(2^j(b-a)+D-1\right)-r(D-1) \\ \vdots \\ r(2)-r\left(2^j(a-b)+2\right) \\ r(1)-r\left(2^j(a-b)+1\right) \end{pmatrix}^T. \qquad (23)$$

The system shown in (22) can be computed when we obtain the column-vector C from the system shown in (15). Then we use least square method to find the inverse of a matrix A. In other words we use pseudo inverse. Denoting $A'=\left(A^T A\right)^{-1}A^T$ as a pseudo inverse of the matrix A[6]. Then column-vector C computed as follows:

$$C=2^{-j/2}f_r A'. \qquad (24)$$

so (22) takes the following form:

$$\int_a^b f(x)dx=2^{-j}Rf_r A'. \qquad (25)$$

Hence Eq. (25) is used to evaluate the values of definite integral of a function using Daubechies' wavelets by linear least square method.

In this section we employ the error analysis for the Daubechies' wavelets bases by linear least square method described above to evaluate an approximation value of definite integral and obtain upper bound of the absolute error.

Lemma1. Let A be $m\times n$ ($m<n$) and full rank matrix. Then the minimum-norm solution x to the underdetermined system $Ax=b$ is given by

$$x=(A^T A)^{(-1)}A^T b, \qquad (26)$$

note that the matrix $(A^T A)^{(-1)}A^T$ is pseudo inverse of A.

proof. The proof is given in [8].

Lemma 2. In Eq. (6), $d_{j,l}\le C\,2^{-j(D+1)/2}\max_{\eta\in I_{j,l}} f^{(p)}(\eta)$, where $C=\frac{1}{p!}\int_0^{D-1}x^p\psi(x)dx$, and $p=D/2$.

Proof. The proof is given in [5].

Lemma 3. Let $f(x)\in L^2(\mathbb{R})$ be a continuous functions difined for $a\le x\le b$, $H=\int_a^b f(x)dx$ and

$H_j = \int_a^b P_{V_j} f(x)dx$ Then:

$$E_j = \left|H - H_j\right| \le C\, 2^{\frac{-j(D+1)}{2}}. \tag{27}$$

Proof. The absolute error may be defined as:

$$E_j = \int_a^b \sum_{i=j}^{\infty} \sum_{l=2^j-D+2}^{2^j b-1} d_{i,l}\psi_{i,l}(x)dx, \tag{28}$$

which $d_{i,l}$ are defined in (7). So

$$\left|E_j\right|^2 = \int_a^b \Big(\sum_{i=j}^{\infty} \sum_{l=2^j-D+2}^{2^j b-1} d_{i,l}\psi_{i,l}(x) \sum_{m=j}^{\infty} \sum_{k=2^j a-D+2}^{2^j b-1} d_{m,k}\psi_{m,k}(x)\Big)dx$$

$$= \sum_{i=j}^{\infty} \sum_{l=2^j-D+2}^{2^j b-1} \sum_{m=j}^{\infty} \sum_{k=2^j a-D+2}^{2^j b-1} d_{i,l} d_{m,k} \int_{-\infty}^{\infty} \psi_{i,l}\psi_{m,k}(x)dx \int_a^b dx$$

$$= \sum_{i=j}^{\infty} \sum_{l=2^j a-D+2}^{2^j b-1} \left|d_{i,l}\right|^2 (b-a) \tag{29}$$

by using lemma 2 in Eq. (27), it takes the form

$$\left|E_j\right|^2 \le \sum_{i=j}^{\infty} \sum_{l=2^j a-D+2}^{2^j b-1} C^2 2^{-i(D+1)} \left(\max_{\eta\in I_{j,l}} f^{(p)}(\eta)\right)^2 (b-a)$$

$$\le \sum_{i=j}^{\infty} C^2 2^{-i(D+1)} \sum_{l=2^j a-D+2}^{2^j b-1} \left(\max_{\eta\in I_{j,l}} f^{(p)}(\eta)\right)^2 (b-a)$$

$$\le C^2 \left(\max_{\eta\in I_{j,l}} f^{(p)}(\eta)\right)^2 \frac{2^{-2j(D+1)/2}}{1-2^{-2(D+1)/2}}, \tag{30}$$

taking the square root of (30) yields

$$\le C\left(\max_{\eta\in I_{j,l}} f^{(p)}(\eta)\right) 2^{-j(D+1)/2}. \tag{31}$$

Hence

$$E_j \le C\, 2^{-j(D+1)/2}. \tag{32}$$

Hence the accuracy in the numerical integration using Daubechies' wavelets bases by linear least square method improves as we increase j or D.

In order to show the numerical result, the following examples are introduced and the approximate value are obtained for j=9,11 and D=4,6,8. In Table 1- 3 the relative errors with respect to the exact value are shown.

Example1.

$$\int_0^1 \sqrt{x^2 - 5x + 12}dx$$

(see Table 1).

Table 1-Relative error above method with respect exact value.

J	D=4	D=6	D=8
9	1.7668E - 07	9.3151E – 08	0.0000E + 00
11	1.1760E - 08	6.1369E - 09	0.0000E + 00

Example2.

$$\int_0^1 \sin(x^2)dx$$

(see Table 2).

Table 2-Relative error above method with respect exact value.

J	D=4	D=6	D=8
9	2.9132E- 06	1.43151E – 06	1.1272E – 09
11	1.2403E - 07	1.10215E - 08	0.0000E + 00

Example3.

$$\int_1^3 \frac{1}{\sqrt{(x-1)^3}}dx$$

(see Table 3).

Table 3-Relative error above method with respect exact value.

J	D=4	D=6	D=8
9	6.7668E - 03	6.3151E – 03	6.1137E - 03
1 1	3.4012E - 03	3.2769E - 03	3.0147E - 03

From Tables 1-3 it can be noticed that above method for numerical integration of definite integrals using Daubechies wavelet by linear least square methods have acceptable accuracy and the approximate value of definite integral is improved by increasing *j* and *D.*

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