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Isogeometric solution of second order ordinary differential equations

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Abstract

In this research, based on the concept of isogeometric analysis, an algorithm is developed for solving second order ordinary differential equations. In isogeometric analysis method, solution of differential equations are considered as imaginary curves or surfaces which are constructed by using advanced versions of splines such as Non-Uniform Rational B-Splines (NURBS). In the same manner, coefficients of differential equations, which themselves might be functions in general, can be assumed as other imaginary curves or surfaces. An IGA framework is created in MATLAB to solve problems. The analysis result is compared with exact and other methods solutions. Finally, the effect of different parameters on the solution of an example is investigated.

Keywords: Isogeometric analysis, NURBS, second order ordinary differential equations

1. INTRODUCTION

Most of problems faced in different disciplines of science and engineering are engaged with solving differential equations. Since only a very limited of these equations can be solved analytically, several numerical methods have been proposed for the solution of differential equations. Amongst the most popular of these methods the finite difference, finite element and the wide range of so called mesh-free methods can be mentioned. The NURBS (non-uniform rational B-splines) based isogeometric analysis method proposed by Hughes et al. in 2005, removes some difficulties of existent methods such as requiring a mesh generation process and inaccurate modeling of the geometry [1].

Isogeometric analysis (IGA) is designed to combine two tasks, design by Computer Aided Design (CAD) and Finite Element Analysis (FEA). Isogeometric analysis is indeed a collection of methods that use splines, or some of their extensions such as NURBS and T-splines, as functions to build approximation spaces which are then used to solve partial differential equations numerically. Due to some interesting properties of splines and NURBS beside accurate definition of geometry, their basis functions can be employed in place of interpolation and approximation functions of finite elements and meshfree methods. From the standpoint of numerical solution of equations, the use of splines and NURBS as building blocks for the construction of discrete spaces, paves the way to many new numerical schemes for the numerical simulations of equations that would be extremely hard to achieve within a standard finite element framework. The smoothness of splines is a new ingredient that yields several advantages: for example, it improves the accuracy per degree of freedom and allows for the direct approximation of equations of order higher than two.

In this paper, based on the concept of isogeometrical analysis, an algorithm is developed for solving second order ordinary differential equations. In this case, the solution might be imagined as a curve which can be generated by using Splines and NURBS. In Section 2, the main concepts of curve definition by Splines is briefly explained. Section 3 is devoted to the derivation of the formulation and the system of equations. In Section 4 the effect of different parameters on the solution of a typical example is investigated. Finally, Conclusions and proposed further research is the subject of Section 5.

2. CURVE DEFINITION BY SPLINES

The formulation of Splines and NURBS can be found in several references and is briefly pointed here [1,2]. Defining a B-spline curve, in its general form, requires the following [3]:

- $(n+1)$ control points P_i, $i=0, 1, \ldots, n$.
- Knot vector U with m components, where $m=n+p+1$.
- Basis functions of degree p.

The B-Spline curve is parametrically constructed as follows:

$$C(u) = \sum_{i=0}^{n} N_{i,p}(u) \times p_i \tag{1}$$

where u is the parameter and $N_{i,p}(u)$ is basis function.

2.1. KNOT VECTOR

The knot vector is defined as $U=\{u_0,u_1,\ldots,u_m\}$ where u_i is a non-decreasing sequence of real numbers; $u_i \leq u_{i+1}$, $i=0,1,\ldots,m$. The u_i is called knot, and U is the knot vector. When, for instance, for every i, we have $u_{i+1} - u_i = u_i - u_{i-1}$ then U is a uniform knot vector and non-uniform vice versa.

2.2. BASIS FUNCTIONS

The i-th B-Spline basis function of degree p (order $p+1$), denoted by $N_{i,p}(u)$, is defined recursively as:

$$N_{i,0}(u) = \begin{cases} 1 & if\ u_i \leq u \leq u_{i+1} \\ 0 & otherwise \end{cases} \tag{2}$$

$$N_{i,p}(u) = \frac{u-u_i}{u_{i+p}-u_i} N_{i,p-1}(u) + \frac{u_{i+p+1}-u}{u_{i+p+1}-u_{i+1}} N_{i+1,p-1}(u) \tag{3}$$

3. DERIVATION OF NUMERICAL FORMULATION

Let's consider the following second order ordinary differential equation to solve.

$$\frac{d}{dx}\left(\alpha(x)\frac{dY(x)}{dx}\right) + \beta(x)Y(x) = f(x) \qquad a \leq x \leq b \tag{4}$$

Now, following a procedure analogous to the isoperimetric finite elements or meshfree methods, the geometrical variables, as well as the unknown function, are approximated by using the spline basis function as below:

$$X(r) = \sum_{i=0}^{n} N_{i,p}(r) \times X_i \tag{5}$$

$$Y(r) = \sum_{i=0}^{n} N_{i,p}(r) \times Y_i \tag{6}$$

Where r is parameter with their values between zero and one. Here, X_i is the x- coordinate of the control points of the solution curve. As it is noted, in the equations above, all of the variables are written in terms of the parameter r, which is similar to mapping in finite elements with the concept of the base or master element. However, calculation of the partial differentials is somehow different and needs special care. With some simple calculus the following relations can be derived.

$$\frac{dY(r)}{dX(r)} = \frac{dY(r)}{dr} \times \frac{dr}{dX(r)} \tag{7}$$

$$\frac{dX(r)}{dr} = \sum_{i=0}^{n} N'_{i,p}(r)X_i = J(r) \tag{8}$$

$$\frac{dY(r)}{dr} = \sum_{i=0}^{n} N'_{i,p}(r)Y_i \tag{9}$$

$$\frac{dY(r)}{dX(r)} = \frac{\sum_{i=0}^{n} N'_{i,p}(r)Y_i}{\sum_{i=0}^{n} N'_{i,p}(r)X_i} = (\sum_{i=0}^{n} N'_{i,p}(r)Y_i)J^{-1}(r) \tag{10}$$

By conversion of (4) into weak form, it follows:

$$\int_a^b w(x)\left[\frac{d}{dx}\left(\alpha(x)\frac{dY(x)}{dx}\right) + \beta(x)Y(x) = f(x)\ \right]dx = 0 \tag{11}$$

$$w(x)\alpha(x)\frac{dY(x)}{dx}\Big|_a^b - \int_a^b \left[\alpha(x)\frac{dw(x)}{dx}\frac{dY(x)}{dx} - \beta(x)w(x)Y(x)\ \right]dx - \int_a^b w(x)f(x)dx = 0 \tag{12}$$

A functional Π can be constructed as [4]:

$$\Pi = \frac{1}{2}B(Y,Y) - l(Y) \tag{13}$$

Where,

$$B(Y,w) = \int_a^b \left[\alpha(x)\frac{dw(x)}{dx}\frac{dY(x)}{dx} - \beta(x)w(x)Y(x)\ \right]dx \tag{14}$$

$$l(w) = w(x)\alpha(x)\frac{dY(x)}{dx}\Big|_a^b - \int_a^b w(x)f(x)dx = 0 \tag{15}$$

By substitution of *w(x)* into *δY* and (10) into (13), it follows:

$$\frac{\partial\Pi}{\partial Y_i} = \int_0^1 [\alpha(r)\big(N'_{i,p}(r)\big(\textstyle\sum_{j=0}^{n} N'_{i,p}(r)Y_i\big)(J^{-1}(r))^2\big) - \beta(r)N_{i,p}(r)\big(\sum_{j=0}^{n} N_{i,p}(r)Y_i\big)\ ]J(r)dr + \int_0^1 N_{i,p}(r)f(r)J(r)dr\ = 0 \tag{16}$$

(16) can be simplified as follows:

$$\begin{bmatrix} \int_{\Omega_0} K_0(r)dr & \cdots & \int_{\Omega_0} K_n(r)dr \\ \vdots & \ddots & \vdots \\ \int_{\Omega_m} K_0(r)dr & \cdots & \int_{\Omega_m} K_n(r)dr \end{bmatrix} \begin{bmatrix} Y_0 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \int_{\Omega_0} F(r)dr \\ \vdots \\ \int_{\Omega_m} F(r)dr \end{bmatrix} \tag{17}$$

4. SOME EXPERIENCES WITH THE METHOD

Example 1: Consider the differential equation:

$$-\frac{d^2u}{dx^2} - u + x^2 = 0, \qquad u(0) = 0, \qquad u'(1) = 1$$

The degree of spline basis functions is taken as *p=2* and a uniform knot vector employed. The number of unknown parameters in isogeometric and other approximate methods are as the same. The isogeometric solution is compared in Table 1 with the exact and four approximate solutions [4]. Subscripts are as follows: RR, Rayleigh-Ritz; PG, Petrov-Galerkin; G, Galerkin; LS, least-squares; C, collocation; IG, Isogeometric.

Table 1- Comparison of the different methods solutions of the problem

x	u_{exact}	u_{RR}	u_{PG}	u_{G}	u_{LS}	u_{C}	u_{IGA}
0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	0.1262	0.1280	0.1285	0.1275	0.1252	0.1348	0.1271
0.2	0.2513	0.2529	0.2536	0.2523	0.2485	0.2668	0.2518
0.3	0.3742	0.3749	0.3754	0.3741	0.3699	0.3958	0.3739
0.4	0.4943	0.4938	0.4941	0.4932	0.4891	0.5216	0.4933
0.5	0.6112	0.6097	0.6096	0.6093	0.6058	0.6440	0.6099
0.6	0.7244	0.7226	0.7221	0.7226	0.7200	0.7628	0.7234
0.7	0.8340	0.8324	0.8317	0.8329	0.8314	0.8778	0.8338
0.8	0.9402	0.9393	0.9384	0.9404	0.9397	0.9887	0.9408
0.9	1.0433	1.0431	1.0424	1.0448	1.0449	1.0954	1.0443
1	1.1442	1.1439	1.1437	1.1463	1.1467	1.1977	1.1442
Error	0.0000	0.0029	0.0039	0.0025	0.0058	0.0500	0.0018

Example 2: Consider the differential equation:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) = 4x(cosx^2 - x^2sinx^2 - 2(cosx^2)^2 + 4x^2cosx^2sinx^2 + 1),\ \ u(0) = 0\,, u(\pi) = -0.8188$$

The exact solution of this problem is known and is given by: *u(x) = sin (1-cos* x^2*)*
The exact and isogeometric solutions, based on different degrees and number of control points, are illustrated in Figures 1-3.

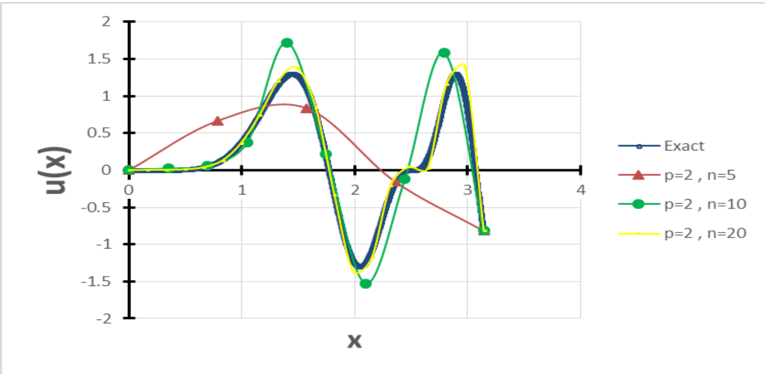

Figure 1. Comparison of the isogeometric and exact solutions of the problem (p=2)

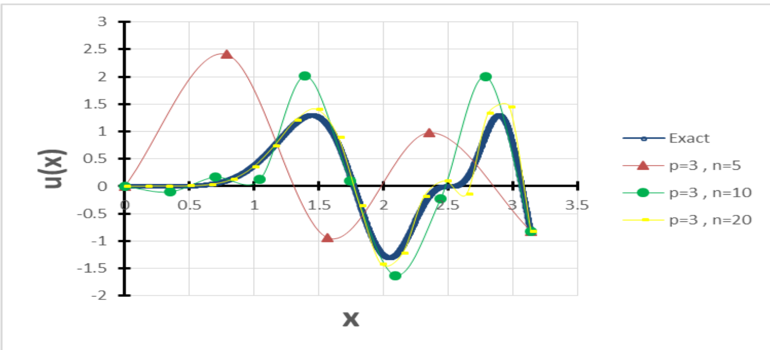

Figure 2. Comparison of the isogeometric and exact solutions of the problem (p=3)

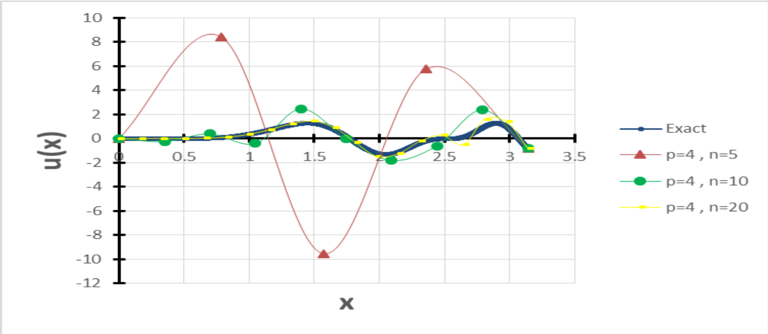

Figure 3. Comparison of the isogeometric and exact solutions of the problem (p=4)

5. CONCLUSIONS

According to this research, it seems that the isogeometric analysis potentially has the capability to substitute the finite element and meshfree methods. When applied to the solution of equation, better results in comparison with the other methods are obtained. Furthermore, the results are not sensitive to the position of control points as well as the knot vectors. Therefore, this method is quite suitable for an adaptive solution and applicable to finite strain problems with geometrical nonlinearity. More research is needed to get a better understanding of the performance of the method in its application to multivariable partial differential equations encountered in different fields of science and engineering.

6. REFERENCES

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