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Modal Analysis of Stepped Timoshenko Shafts by the Finite Element Method

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Abstract
In this paper, free vibration analysis of a stepped Timoshenko shaft with negligible damping is investigated by means of the finite element method in both uniform and non-uniform status. Non-uniformity is assumed to be in cross sectional area of the shaft where the middle part of the stepped shaft is in conical form. The Timoshenko beam element with linear shape functions was used for the finite element modeling of the shaft. Assembling of mass and stiffness matrices, equation of motion derivation and eigenvalue solution were performed by use of MATLAB software. Natural frequencies of the shaft were calculated and their corresponding mode shapes were plotted for two boundary conditions namely clamped and simply supported. The difference between natural frequencies obtained from the Timoshenko model were compared with those from classical Bernoulli-Euler theory and the effects of rotary inertia and shear deformation on the magnitudes of calculated natural frequencies were studied. It has been observed that uniform stepped shafts are stiffer than non-uniform stepped ones for equal amount of mass, since their frequency parameter is higher. Additionally, it became evident that the effects of transverse shear and rotary inertia will be much more pronounced as the mode number increases.
Keywords: *Timoshenko-FEM-Classic Beam Theory-Stepped Shaft.*

Introduction

It is for decades that the finite element method has been used in many engineering areas such as civil, mechanical and aerospace. With the advent of digital computers it has got an immense amount of interest since the capability of computers to perform huge amount of processes in a fraction of second made them very suitable for such a method to stabilize its base on an imperturbable firm. Some researchers used analytical methods to solve the problems but as these methods are limited to simple geometry and other simplifications in the main physical problem, they have been forced to use different numerical and approximate methods in order to consider different geometries in their studies.

Vibration analysis of beams and shafts with different characteristics and different geometries is a topic of engineering analysis particularly aerospace engineering for a long time [1]. Vibration and dynamic analysis of ribs and spars and other types of beam-like structures, is always a challenging area in aerospace structures like aircrafts and spacecrafts.

From the most substantial theories of Bernoulli, Euler, Timoshenko and other great scientists of their time to the most challenging and new theories for analysis of nano-beams and micro-cantilevers used in MEMS and NEMS in these days, it can be implied the importance and huge amount of studies conducted on this topic. Thanks to the theory formulated by Timoshenko we can account for the effects of rotary inertia and shear deformation in vibration of thick beams. Since the classical Bernoulli-Euler theory assumes the shear rigidity of the beam to be infinite it is useful for slender beams vibrating in just their first natural frequencies.

In [3] Sarigul and Aksu studied the free vibration of stepped Timoshenko shafts and beams using finite difference method and compare their results with those from Bernoulli-Euler theory. Non-linear free vibrations of stepped thickness beams were studied by Sato in [9] where he accounts for the effects of nonlinearity in his study. A piecewise continuous Timoshenko beam model for the dynamic analysis of tapered beam-like structures was proposed by Shen et al in [11]. In [14] Rao et al provided a finite element formulation for large amplitude free vibrations of beams and orthotropic circular plates. They studied the effects of variable cross section on the response of structures to various kinds of loading. In [18] Lou et al applying third order shape functions to the nodal displacements, studied the dynamic behavior of a uniform Timoshenko beam under moving concentrated loads with the finite element method.

In this article the model to describe the shaft was based on Timoshenko theory which does account for the effects of rotary inertia and shear deformation in addition to the assumptions of classical Bernoulli-Euler theory. Here, we had studied free vibration of stepped uniform and non-uniform Timoshenko shafts with two different boundary conditions. Utilizing linear shape functions for the Timoshenko beam elements we derive the element mass and stiffness matrices. Assembling the element matrices will yield the global mass and stiffness matrices and then the

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equation of motion. By solving the eigenvalue problem resulted from the assumptions above, natural frequencies will be calculated and consequently corresponding mode shapes can be plotted for shafts subjected to different boundary conditions.

Model of a Timoshenko Beam Element

A simply supported beam is shown in Fig. 1. A coordinate system is assumed to be fixed in the inertial frame, with the x-axis parallel to the undeformed longitudinal axis of the beam and the y-axis pointing vertically downward in the same direction as the gravitational acceleration g. It is assumed that the downward displacement of the Timoshenko beam is taken as positive and that it is measured with reference to its vertical static equilibrium position.

Fig. 1: A Simply Supported Uniform Beam

The Timoshenko beam is discretized into a number of simple elements with equal length. Fig. 2 shows a Timoshenko beam element of length l. The beam element consists of two nodes i and j; each node has two degrees of freedom, i.e., vertical displacement w and bending rotation (or slope) $\emptyset$. The vertical displacement w and bending rotation $\emptyset$ of an arbitrary point on the beam element can be expressed as:

$$w = [N_i\ 0\ N_j\ 0].\{q\}^e \quad (1)$$

$$\emptyset - [0\ N_i\ 0\ N_j\ 0].\{q\}^e \quad (2)$$

in which $\{q\}^e$ is the element nodal displacement vector and the shape functions are written as :

$$N_i = \frac{1}{x_{ji}}(x_j - x) \quad (3)$$

$$N_j = \frac{1}{x_{ji}}(x - x_i) \quad (4)$$

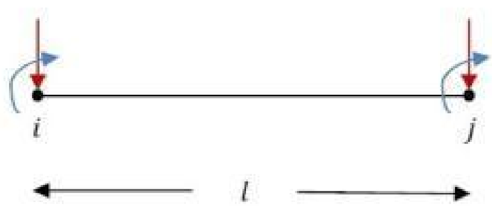

Fig. 2: Timoshenko Element

The virtual work of a Timoshenko beam element and the equation of motion

According to Fig. 2, the virtual work of this beam element consists of the internal virtual work δW_I^e and the external virtual work δW_E^e. It is assumed that the damping effect of the beam is neglected, the internal virtual work δW_I^e of this beam element can be written as:

$$\delta W_I^e = \int_0^l EI\left(\frac{\partial\emptyset}{\partial x}\right).\delta\left(\frac{\partial\emptyset}{\partial x}\right)dx + \int_0^l kAG\left(\frac{\partial w}{\partial x} - \emptyset\right)\delta\left(\frac{\partial w}{\partial x} - \emptyset\right)dx \quad (5)$$

The external virtual work δW_E^e of this beam element can be expressed as :

$$\delta W_E^e = -\int_0^l \rho A\ddot{w}^e.\delta w^e dx - \int_0^l \rho I\ddot{\emptyset}^e.\delta\emptyset^e dx \quad (6)$$

where ρ is the mass density of the beam material; and the dot above the symbol denotes the differentiation with respect to time t. Performing the differentiation with respect to coordinate x and time t, stiffness and mass matrices for the beam element will be derived as:

consistent mass matrix for translational inertia =
$[M_t]^e = \int_0^l \rho A[N_w]^T[N_w]dx$

consistent mass matrix for rotatory inertia =
$[M_r]^e = \int_0^l \rho I[N_\emptyset]^T[N_\emptyset]dx$

bending stiffness matrix = $[K_b]^e = \int_0^l EI[N_\emptyset']^T[N_\emptyset']dx$

shear stiffness matrix =
$[K_s]^e = \int_0^l kAG([N_w']^T - [N_\emptyset]^T)([N_w'] - [N_\emptyset])dx$

through which the equation of motion for the beam element can be derived as

$$[M]^e\{\ddot{q}\}^e + [K]^e\{q\}^e = 0 \quad (7)$$

where

$$[M]^e = [M_t]^e + [M_r]^e$$

$$[K]^e = [K_b]^e + [K_s]^e$$

By assembling element matrices and element nodal vectors, respectively, one can obtain the global equation of motion for a Timoshenko beam neglecting the damping, which will appear as

$$[M]\{\ddot{q}\} + [K]\{q\} = 0 \quad (8)$$

where the matrices $[M]$ and $[K]$ are the global mass and stiffness matrices, respectively, of the Timoshenko beam; the vectors $\{\ddot{q}\}$ and $\{q\}$ are the nodal acceleration and displacement vectors, respectively, of the beam. Eq. (8) could successfully analyze the free vibration response of an elastic Timoshenko beam with various boundary conditions.

Results and Discussion

In this section, as shown by Fig. 3, two types of stepped shafts have been considered; namely uniform and non-uniform. In this problem, E=30*10e6 and G=90/8*10e6 psi are used as Young's modulus and shear modulus. The value of shear coefficient was set to k=0.886 as proposed by Cowper [21] for a circular cross-section having a Poisson ratio equal to 0.3. The material of the shaft is steel with mass density of 0.28 and length of 10 inches.

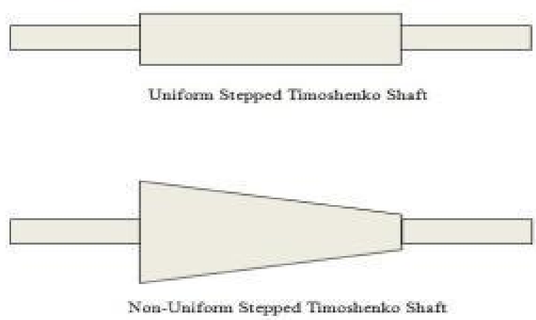

Fig. 3: Typical Timoshenko shaft

In all computations the number of elements has been set to 120 because of good convergence and fine running time considerations. In order to verify the validity of the method we have performed a comparison for the natural frequencies and mode shapes with [3]. Results for the frequencies are summerized in Tables 1 to 3 and for mode shapes are shown in Figs. 4 to 7. The comparison of the natural frequency parameters for Timoshenko and Bernoulli-Euler shafts are summarized in Table 1.

Table 1: Timoshenko vs Bernoulli-Euler dimensionless natural frequency parameters for uniform simply supported stepped shaft

Mode Number	Present Method	Bernoulli-Euler	Discrepancy (%)	Omega Ratio
1	5.4764	5.6535	3.13	0.96
2	16.7749	18.695	10.27	0.89
3	47.3561	67.8684	30.22	0.69
4	72.3755	117.1585	38.22	0.61
5	92.5306	151.0257	38.73	0.61

We can observe that the predicted natural frequencies by Bernoulli-Euler theory are higher than those predicted by Timoshenko theory since the Bernoulli-Euler theory do not consider the shear deformation and assumes that the shear rigidity is infinite so the predicted frequencies will be slightly higher. Another observation is that as the mode number increases the difference between two theories are more sensible because the effects of rotary inertia and shear deformation are functions of the wave number hence increasing the mode number will increase the difference between two theories. The comparison between the present method and the method of Sarigul [3] has been shown in Tables 2 and 3.

Table 2: Dimensionless frequency parameters for Uniform Timoshenko shaft with simply-supported BC

Mode Number	Present Method	Sarigul	Discrepancy (%)
1	5.4764	5.6256	2.65
2	16.7749	17.5545	4.44
3	47.3561	55.8303	15.17
4	72.3755	88.4739	18.19
5	92.5306	109.4663	15.47

Table 3: Dimensionless frequency parameters for Non-Uniform Timoshenko shaft with simply-supported BC

Mode Number	Present Method	Sarigul	Discrepancy (%)
1	5.3838	5.461	1.41
2	16.4799	17.3431	4.97
3	48.8949	54.5734	10.4
4	71.8914	88.25	18.53
5	90.8724	108.3535	16.13

We can observe from the results for frequency parameter that the method presented here estimates natural frequencies in a reasonable way since bringing into consideration the effects of shear deformation the calculated natural frequencies seems to be lower according to the fact that shear rigidity is not infinite. The difference between results herein and the results by Sarigul [3] are more pronounced in higher modes.

According to Tables 1 to 3 the frequency parameters in non-uniform shafts are lower than those in uniform shafts so we can conclude that for same mass and geometric conditions the rigidity of uniform shaft is higher than that of non-uniform one.

Next, you can see the corresponding mode shapes in Figs 4 to 7. General shape of the shafts vibrating in their associated natural frequency is plotted versus the number of elements along the shaft length. For the purpose of spatial limitations, just two modes of vibration for two different boundary conditions namely simply supported and clamped shafts are presented here. In these figures, the red curves are for uniform shaft and blue ones are for non-uniform shaft.

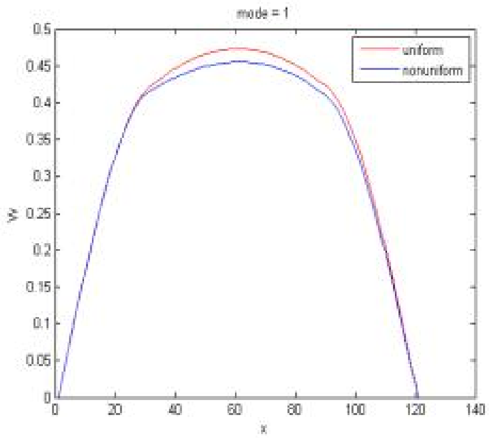

Fig. 4: Displacement Mode No.1 for simply supported shaft

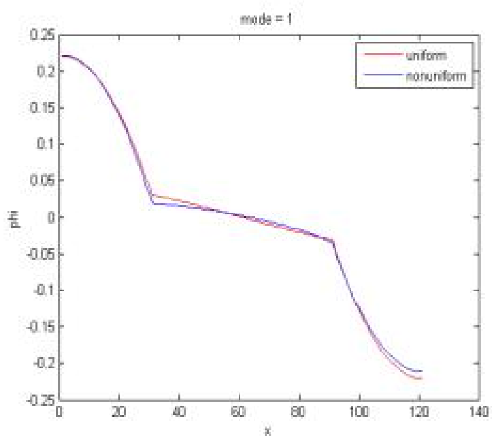

Fig. 5: Rotation Mode No. 1 for simply supported shaft

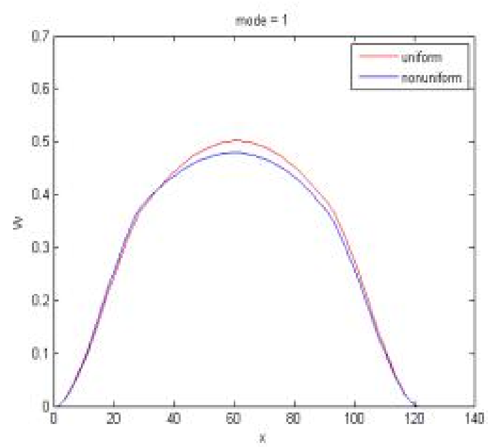

Fig. 6: Displacement Mode No.1 for clamped shaft

The difference between the locations of the corners of the curves in Fig. 4 is due to the steps and non-uniformity of the shaft. The middle part of the shaft is tapered and the diameter of the shaft at the first step is more than that of the shaft at the second step so we could verify the changes at the corners of the curves in Fig. 4.

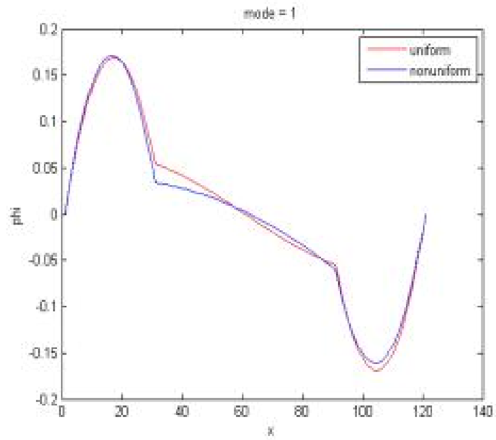

Fig. 7: Rotation Mode No. 1 for clamped shaft

From Figs 4 to 7 we can see the difference between the simply supported and clamped boundary conditions in a way that for clamped boundary condition the curves for vertical displacement at the beginning and the end of the shaft has the zero value for slope or rotation, while in simply supported boundary condition there is no zero slope. This can be seen clearly in Figs 5 and 7 where the slope at the two ends of the shaft for clamped shaft is zero but for the simply supported shaft it has non-zero value. Also it is observed that the effects of non-uniformity in the cross section of the shaft showed itself as large and irregular amplitudes in comparison to uniform shaft.

Conclusions

One of the major contributions of the method is that different boundary conditions can be considered in the analysis without any difficulty in calculation of the results. In the analysis of stepped uniform and non-uniform shafts it has been observed that uniform stepped shafts are stiffer than non-uniform stepped ones for equal amount of mass, since their frequency parameter is higher. The difference between mode shapes of uniform and non-uniform stepped shafts are more marked for thick shafts. The effects of transverse shear and rotary inertia increase as the mode number increases. In higher modes, the difference between mode shapes of uniform and non-uniform shafts is pronounced more and the instability in non-uniform shaft between the two steps is somehow more compared to the uniform shaft.

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