Vibration analysis of rotating disk considering radial and circumferential in-plane stresses created by rotation

Keivan Torabi [1], Zohre Biyabani [2], Hassan Afshari[3], Alireza Pouretemad[4]

1,3,4- Solid Mechanics Department, Faculty of Mechanical Engineering, University of Kashan, Kashan, Iran.

2- Faculty of Mechanical Engineering, Islamic Azad University, Khomeinishahr Branch, Isfahan, Iran

Abstract

In this paper, generalized differential quadrature method (GDQM) is employed to analyze vibration of a rotating disks by considering effect of radial and circumferential in-plane stresses. Natural frequencies and vibration modes are derived numerically. First, the versatility and accuracy of the presented solution are tested against presented exact results. Then, effects of the ratio of radii, rotation speed and circumferential mode number on the natural frequencies are investigated.

Keywords: *"Generalized differential quadrature method (GDQM)", "Free vibration", "Rotating disk".*

Introduction

The determination of the dynamic response of rotating disks, i.e., the mode shapes and natural frequencies, is an important prerequisite in design of rotating equipments. Rotating annular disks are widely used in mechanical and aerospace engineering such as circular saws, turbines, flywheels, CDs and DVDs in data storage, and so on. Many researchers have studied the vibration of annular plates during the long period of time. An excellent survey of the old researches on the free vibration analysis of annular plates has been done by Leissa [1].

Because of created in-plane stresses, rotation of disks has a significant effect on their natural frequencies; The first analyses of rotating disks was performed by Lamb and Southwell [2,3], Barasch and Chen [4] and Simmonds [5,6] which the rotating disks were modeled by rotating membranes. A complete solution of the fully clamped rotating membrane was finally presented by Eversman and Dodson [7,8]. Approximate techniques, have also been used in particular by Mote [9] to study the free vibration characteristics of initially stressed, fully clamped, variable thickness disks operating in a prescribed thermal environment. Harish [10] presented a perturbation analysis of a rotating annulus clamped at the hub and free at the outer edge. In addition, Irie et al. [11] studied stress distributions and flexural vibration of rotating annular discs with radially varying thickness by means of a spline interpolation technique. Using perturbation technique, Mignolet et al. [12] derived natural frequencies and mode shapes of a fexible rotating disk clamped at the hub and free at the outer edge

In this paper, GDQM is applied to analyze free vibration of annular plates. Natural frequencies and vibration modes are obtained and compared with the published results of other researchers. Comparison of the present and previous results confirms the convergence and accuracy of the proposed solution. Effects of the ratio of radii, rotation speed and circumferential mode number on the natural frequencies are investigated and discussed.

Vibration analysis

As depicted in Fig. 1, a uniform disk rotating at constant angular velocity Ω is considered. The governing equation for free vibration has been expressed as [11]

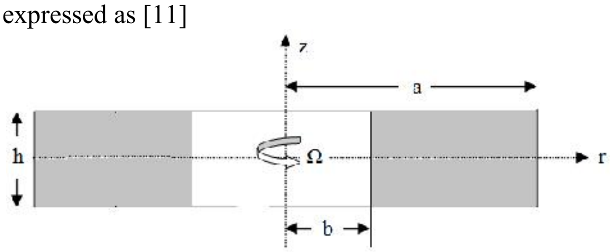

Fig. 1: Geometry and parameters of rotating annular plate.

$$D\begin{pmatrix} \frac{\partial^4 w}{\partial r^4}+\frac{2}{r}\frac{\partial^3 w}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} \\ +\frac{2}{r^2}\frac{\partial^4 w}{\partial r^2\partial\theta^2}+\frac{1}{r^3}\frac{\partial w}{\partial r} \\ -\frac{2}{r^3}\frac{\partial^3 w}{\partial r\partial\theta^2}+\frac{4}{r^4}\frac{\partial^2 w}{\partial\theta^2}+\frac{1}{r^4}\frac{\partial^4 w}{\partial\theta^4} \end{pmatrix} -\frac{1}{r}\frac{\partial}{\partial r}\left(rh\sigma_r\frac{\partial w}{\partial r}\right)-\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(h\sigma_\theta\frac{\partial w}{\partial\theta}\right)+\rho h\frac{\partial^2 w}{\partial t^2}=0, \quad (1)$$

where w(r,θ,t), ρ and ν are transverse deflection, mass per unit volume and Poisson's ratio, respectively; D is the flexural rigidity of the plate defined as

$$D=\frac{Eh^3}{12(1-\nu^2)}, \quad (2)$$

in which E is modulus of elasticity of material and σ_r and σ_θ are radial and circumferential in-plane stresses created by rotation of the disk presented as [12]

$$\sigma_r=\frac{\mu_2}{r^2}\left(a^2-r^2\right)\left(r^2+\frac{\mu_1}{\mu_2 a^2}\right)$$
$$\sigma_\theta=\frac{\mu_2}{r^2}\left[\left(a^2-\frac{\mu_1}{\mu_2 a^2}\right)r^2-\frac{\mu_1}{\mu_2}-\frac{1+3\upsilon}{3+\upsilon}r^4\right], \quad (3)$$

where

$$\mu_1=\frac{(1-\upsilon)\rho a^2 b^2\Omega^2}{8}\left[\frac{(3+\upsilon)a^2-(1+\upsilon)b^2}{(1+\upsilon)a^2+(1-\upsilon)b^2}\right].$$
$$\mu_2=\frac{(3+\upsilon)\rho\Omega^2}{8} \quad (4)$$

The boundary conditions should reflect the clamp at the hub and the free condition at the outer edge. At the hub the displacement and slope must vanish,

1. Assotiated Professor
2. Grajuated Msc Student
3. Ph.D.Student, Tel: +983655912448, Fax: +983655559930, afshari_hasan@yahoo.com (corresponding author)
4. Ph.D.Student

while, at the free edge, both the moment and the shear should be set to zero. This is, at r=b,

$$w=0 \quad \frac{\partial w}{\partial r}=0 \tag{5}$$

and at r=a [11]

$$M_r=-D\left(\frac{\partial^2 w}{\partial r^2}+\frac{\upsilon}{r}\frac{\partial w}{\partial r}+\frac{\upsilon}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)=0,$$
$$V_r=-D\begin{pmatrix}\frac{\partial^3 w}{\partial r^3}+\frac{1}{r}\frac{\partial^2 w}{\partial r^2}+\frac{2-\upsilon}{r^2}\frac{\partial^3 w}{\partial r\partial\theta^2}\\-\frac{1}{r^2}\frac{\partial w}{\partial r}-\frac{3-\upsilon}{r^3}\frac{\partial^2 w}{\partial \theta^2}\end{pmatrix}=0. \tag{6}$$

Using following dimensionless parameters:

$$\zeta=\frac{r}{a}\quad \varphi=\frac{b}{a}\quad \gamma^2=\frac{\rho h a^4\Omega^2}{D}\quad \lambda^2=\frac{\rho h a^4\omega^2}{D}$$
$$f_r(\zeta)=\frac{3+\upsilon}{8}\left(1-\zeta^2\right)\left(1+\varphi^2\frac{1-\left(\frac{1+\upsilon}{3+\upsilon}\right)\varphi^2}{\frac{1+\upsilon}{1-\upsilon}+\varphi^2}\frac{1}{\zeta^2}\right),$$
$$f_\theta(\zeta)=\frac{3+\upsilon}{8}\left[1-\varphi^2\frac{1-\left(\frac{1+\upsilon}{3+\upsilon}\right)\varphi^2}{\frac{1+\upsilon}{1-\upsilon}+\varphi^2}\left(1+\frac{1}{\zeta^2}\right)-\frac{1+3\upsilon}{3+\upsilon}\zeta^2\right] \tag{7}$$

and applying method of separation of variables as

$$w(\zeta,\theta,t)=e^{i\omega t}\sum_{k=0}^{\infty}W_k(\zeta)\cos(k\theta), \tag{8}$$

the set of governing equations can be rewritten in the following dimensionless form:

$$\frac{d^4W_k}{d\zeta^4}+\frac{2}{\zeta}\frac{d^3W_k}{d\zeta^3}-\frac{1+2k^2}{\zeta^2}\frac{d^2W_k}{d\zeta^2}+\frac{1+2k^2}{\zeta^3}\frac{dW_k}{d\zeta}+\frac{k^4-4k^2}{\zeta^4}W_k$$
$$-\gamma^2\begin{Bmatrix}f_r(\zeta)\frac{d^2W_k}{d\zeta^2}+\left[\frac{df_r(\zeta)}{d\zeta}+\frac{f_r(\zeta)}{\zeta}\right]\frac{dW_k}{d\zeta}\\-\frac{k^2f_\theta(\zeta)}{\zeta^2}W_k\end{Bmatrix}-\lambda^2W_k=0. \tag{9}$$

It should be noticed that in Eq. (8), ω is circular natural frequency of vibration and in Eq. (7).

By substituting Eq. (8) into the Eq. (6), radial component of bending moment and effective shear force can be written in the new following form:

$$M_r(\zeta,\theta,t)=e^{i\omega t}\sum_{k=0}^{K}M_k^r(\zeta)\cos(k\theta)$$
$$V_r(\zeta,\theta,t)=e^{i\omega t}\sum_{k=0}^{K}V_k^r(\zeta)\cos(k\theta) \tag{10}$$

where

$$M_r=-\frac{D}{a^2}\left(\frac{d^2W_k}{d\zeta^2}+\frac{\upsilon}{\zeta}\frac{dW_k}{d\zeta}-\frac{\upsilon k^2}{\zeta^2}W_k\right),$$
$$V_r=-\frac{D}{a^3}\begin{bmatrix}\frac{d^3W_k}{d\zeta^3}+\frac{1}{\zeta}\frac{d^2W_k}{d\zeta^2}-\\\frac{1+(2-\upsilon)k^2}{\zeta^2}\frac{dW_k}{d\zeta}+\frac{(3-\upsilon)k^2}{\zeta^3}W_k\end{bmatrix}, \tag{11}$$

and therefore, dimensionless form of boundary conditions can be written as

$$W_k(\varphi)=0\quad \left.\frac{dW_k}{d\zeta}\right|_{\zeta=\varphi}=0$$
$$\left.\left(\frac{d^2W_k}{d\zeta^2}+\upsilon\frac{dW_k}{d\zeta}-\upsilon k^2W_k\right)\right|_{\zeta=1}=0\quad .$$
$$\left.\begin{bmatrix}\frac{d^3W_k}{d\zeta^3}+\frac{d^2W_k}{d\zeta^2}\\-\left[1+(2-\upsilon)k^2\right]\frac{dW_k}{d\zeta}+(3-\upsilon)k^2W_k\end{bmatrix}\right|_{\zeta=1}=0 \tag{12}$$

The Differential Quadrature Method

Differential quadrature method (DQM) is based on this idea that all derivatives of a function can be approximated by means of the weighted linear sum of the functions values at N pre-selected grid of discrete points as

$$\left.\frac{d^sf}{d\zeta^s}\right|_{r=r_i}=\sum_{j=1}^{N}A_{ij}^{(s)}f_j, \tag{13}$$

where A(s) is the weighting coefficient associated with the s-th order derivative. This matrix is given as [13]

$$A_{ij}^{(1)}=\begin{cases}\dfrac{\prod_{m=1,\,m\neq i,j}^{N}(\zeta_i-\zeta_m)}{\prod_{m=1,\,m\neq j}^{N}(\zeta_j-\zeta_m)}, & (i,j=1,2,3,...,N;i\neq j)\\ \sum_{m=1,\,m\neq i}^{N}\dfrac{1}{(\zeta_i-\zeta_m)}, & (i=j=1,2,3,...,N)\end{cases} \quad . \tag{14}$$
$$A^{(s)}=A^{(1)}A^{(s-1)}\quad s=2,3,\ldots,N-1$$

Distribution of grid points plays an important role in convergence of the solution. A well-accepted set of the grid points is the Gauss–Lobatto–Chebyshev points given for [0,1] as

$$\zeta_i=\frac{1}{2}\left\{1-\cos[\frac{(i-1)\pi}{(N-1)}]\right\},\ (i=1,2,3,...,N). \tag{15}$$

The differential quadrature analogue

In order to simplify in notations, weighting coefficients matrices associated with the first four derivative will be shown respectively as

$$A=A^{(1)}\quad B=A^{(2)}\quad C=A^{(3)}\quad D=A^{(4)}, \tag{16}$$

Using DQ rules, governing equation (9) takes the following form:

$$[R_k]\{W_k\}=\lambda^2\{W_k\}, \tag{17}$$

where

$$[R_k]=\begin{Bmatrix}[D]+2[s][C]-\left(1+2k^2\right)[s]^2[B]\\+\left(1+2k^2\right)[s]^3[A]+\left(k^4-4k^2\right)[s]^4\end{Bmatrix}$$
$$-\gamma^2\left\{[a][B]+[b][A]-k^2[c]\right\}. \tag{18}$$

In equation (19), the following diagonal matrices are defined:

$$a_{ij}=f_r(\zeta_i)\delta_{ij}\quad b_{ij}=\left.\left(\frac{df_r(\zeta)}{d\zeta}+\frac{f_r(\zeta)}{\zeta}\right)\right|_{\zeta=\zeta_i}\delta_{ij}\quad .$$
$$c_{ij}=\frac{f_\theta(\zeta_i)}{\zeta_i^2}\delta_{ij}\qquad s_{ij}=\frac{1}{\zeta_i}\delta_{ij} \tag{19}$$

The grid points can be divided in two groups as boundary points (b) and domain ones (d) considered as

$$\{W_k\}_b = \begin{Bmatrix} W_1 \\ W_2 \\ W_{N-1} \\ W_N \end{Bmatrix} \quad \{W_k\}_d = \begin{Bmatrix} W_3 \\ \vdots \\ W_{N-2} \end{Bmatrix}. \tag{20}$$

The governing equation should be satisfied only for domain points [14]. Thus

$$\left[\bar{R}_k\right]\{W_k\} = \lambda^2 \{W_k\}_d \,. \tag{21}$$

Eq. (22) may be rearranged and partitioned as

$$\left[\bar{R}_k\right]_b \{W_k\}_b + \left[\bar{R}_k\right]_d \{W_k\}_d = \lambda^2 \{W_k\}_d \,. \tag{22}$$

Also, DQ form of boundary conditions can be written as

$$[S_k]\{W_k\} = 0, \tag{23}$$

where

$$\begin{gathered} [S_k]_{1j} = \delta_{1j} \quad [S_k]_{2j} = A_{1j} \\ [S_k]_{3j} = B_{Nj} + \upsilon A_{Nj} - \upsilon k^2 \delta_{Nj} \\ [S_k]_{3j} = \begin{bmatrix} C_{Nj} + B_{Nj} - \left[1 + (2-\upsilon)k^2\right] A_{Nj} \\ + k^2 (3-\upsilon)\delta_{Nj} \end{bmatrix} \end{gathered} \tag{24}$$

Again, Eq. (24) may be rearranged and partitioned as

$$[S_k]_b \{W_k\}_b + [S_k]_d \{W_k\}_d = \{0\}. \tag{25}$$

Using Eqs (23) and (26), the following eigenvalue problem will be achieved:

$$[K_k]\begin{Bmatrix} \{W_k\}_b \\ \{W_k\}_d \end{Bmatrix} = \lambda^2 [M_k] \begin{Bmatrix} \{W_k\}_b \\ \{W_k\}_d \end{Bmatrix}, \tag{26}$$

where

$$[K_k] = \begin{bmatrix} \left[\bar{R}_k\right]_b & \left[\bar{R}_k\right]_d \\ [S_k]_b & [S_k]_d \end{bmatrix} \quad [M_k] = \begin{bmatrix} [0] & I \\ [0] & [0] \end{bmatrix}. \tag{27}$$

Using Eq. (27), natural frequencies and mode shapes can be derived. It should be mentioned that number of grid points should be considered to satisfy the following equation for convergence of first n frequencies:

$$\left| \frac{\lambda_l^{(N)} - \lambda_l^{(N-1)}}{\lambda_l^{(N-1)}} \right| \le \varepsilon \quad l = 1, 2, \ldots, n \,, \tag{28}$$

Numerical results and discussion

Using GDQM, free vibration analysis of rotating disk was proposed. In this section numerical results are presented for various cases. In this paper for all numerical cases, ε is considered as 0.01.

In order to check the accuracy of the presented solution, a stationary uniform disk is considered. Table 1, shows value of the first frequency for k=0,1,2,3,4 (λ_{10}- λ_{14}). According to the results of this table, the presented results are in excellent agreement with the reported results by Zhou et al. [5]. Also, corresponding modes are depicted in Fig. 2.

Table 1: Dimensionless frequency for the non-rotating uniform disk (υ=1/3, φ=0.2).

	λ_{10}	λ_{11}	λ_{12}	λ_{13}	λ_{14}
Present	5.2028	4.8074	6.3601	12.3557	21.5700
Zhou et al. [5]	5.2135	4.8171	6.3431	12.395	21.233

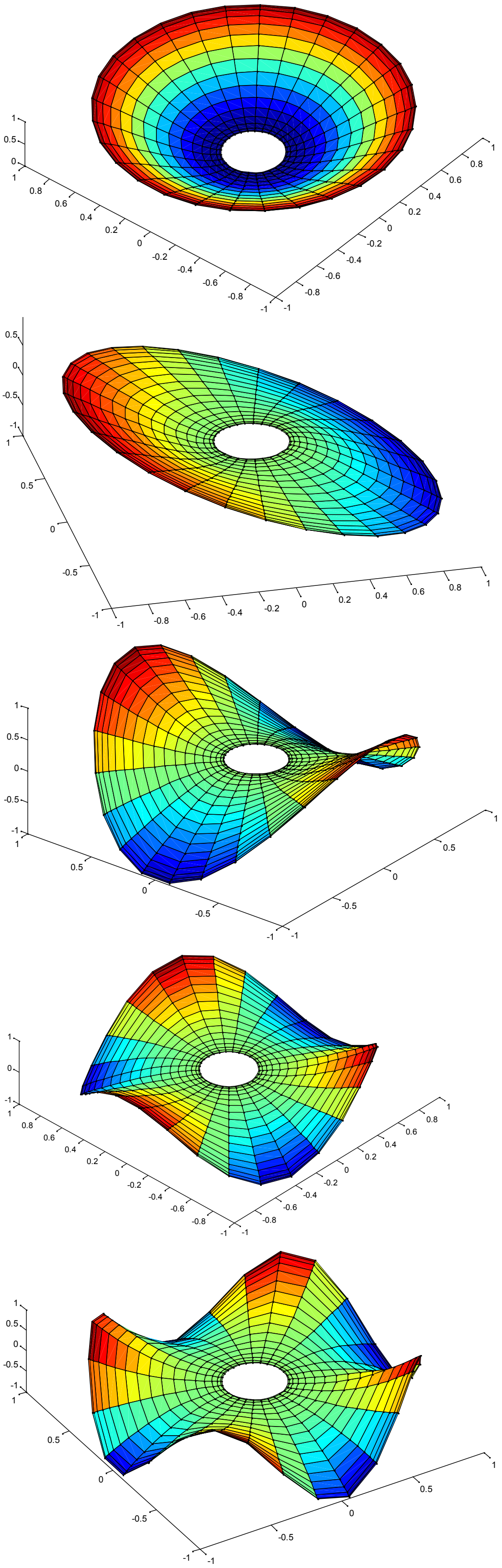

Fig. 2: Vibration modes of non-rotating uniform annular plate ($\lambda_{10} - \lambda_{14}$).

In order to study the effect of the rotation on the natural frequencies, a rotating disk (ν=0.3, φ=0.25) is considered. Figs. 3 shows variation of λ_{10}-λ_{33} versus dimensionless rotating speed. As these figures show all frequencies grow as value of the rotating speed increases. It can be explained by increasing in the stiffness of the disk. Fig. 3 also confirms that natural frequencies increase for higher values of the circumferential mode number

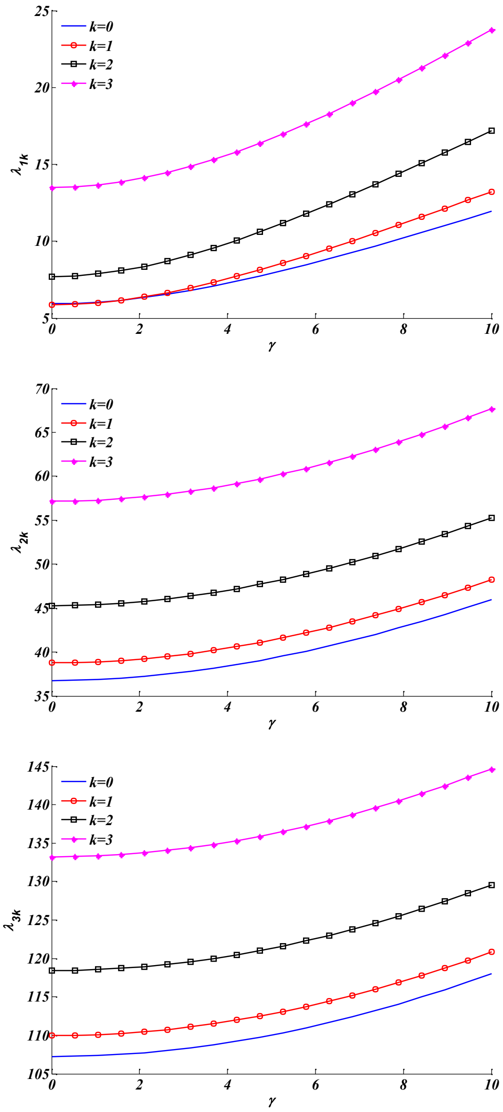

Fig. 3: Variation of λ_{10}- λ_{33} versus dimensionless rotating speed for a disk with exponential change in thickness.

In order to investigate the effect of the ratio of radii (φ) on the natural frequencies, a rotating disk (ν=0.3, γ=10) is considered; Fig. 4 shows variation of λ_{10}-λ_{33} versus ratio of radii. As shown in these figures, all frequencies increase as value of ratio of radii grows. It can be explained by decreasing in the mass of the disk. Fig. 4 also shows that effect of the circumferential mode number (k) on the natural frequencies decreases at higher modes.

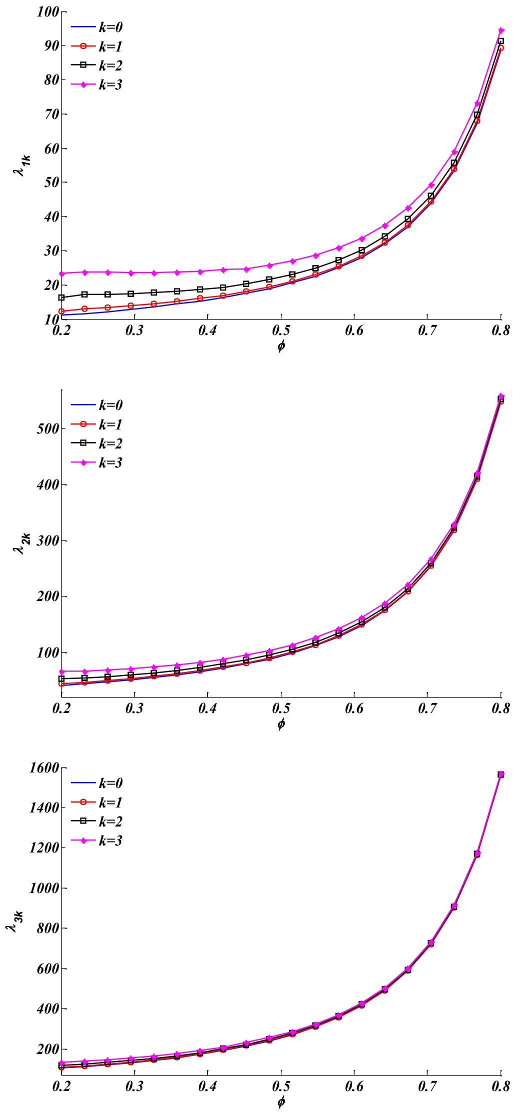

Fig. 4: Variation of λ_{10}- λ_{33} versus ratio of radii for a disk with parabolic change in thickness.

Conclusion

In this paper, by considering effect of radial and circumferential in-plane stresses, free vibration analysis of rotating disks was presented. Natural frequencies and vibration modes were obtained. The accuracy of the presented solution was tested against previous results for vibration analysis of non-rotating annular plates, and the effect of the ratio of radii and rotation speed on the natural frequencies were investigated. Numerical results showed that all frequencies increase as values of the rotating speed, ratio of radii and circumferential mode number increase.

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