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Optimal Line-of-Sight Guidance Law for Moving Targets

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Abstract
In this paper, analytical solutions of a class of optimal line-of-sight (LOS) guidance laws are derived for moving targets using linearized model. The pursuer control dynamics is assumed to be perfect and the maneuvering acceleration is applied normal to LOS. The closed-loop optimal solutions are obtained for four different final conditions, i.e., fixed or free final position and normal velocity. The final angle constraint may be applied by the final normal velocity.

Keywords: *Optimal Guidance – Line of Sight Trajectory – Three Point Guidance.*

Introduction

In three-point guidance, a pursuer maneuvers so as to be on the line-of-sight (LOS) between the target tracker (or reference point) and the target. This guidance law is also called LOS guidance [1-4]. Two implementations of LOS guidance are used in practice, namely, Command-to-LOS (CLOS) and beam riding. Beam riding performance can be significantly improved by taking the beam motion into account in a CLOS system [1-4].

Most literature on LOS guidance dealt with the design of controllers with different control techniques, such as classical control [5,6], feedback linearization [7], variable structure [8], fuzzy-logic [9-11], and optimal control [12-14].

Analytical solutions of LOS guidance problems are more difficult than two-point guidance problems, because of the trajectory constraint, so only simple cases have been obtained in closed-form [15,16]. Closed-loop solutions of optimal three-point guidance laws are also more complex than the two-point strategies. The optimal solution for perfect control system has been obtained in Ref. [13] for stationary targets. The optimal solution for the first-order control system has been presented in Ref. [14] for stationary targets, as well. To the author's knowledge, the available closed-loop optimal LOS solutions have been developed only for stationary targets.

In this work, the optimal solutions of LOS guidance laws are derived for moving targets with different final conditions using linearized model.

Linearized Formulation

Consider pursuer P and its target in polar coordinates (r,θ). The origin of the polar coordinates is located on the tracker as a reference point. The governing equations of motion of particle P in polar coordinates are given by

$$\ddot r_p - r_p\dot\theta_p^2 = a_{p_r} \tag{1a}$$

$$r_p\ddot\theta_p + 2\dot r_p\dot\theta_p = a_{p_\theta} \tag{1b}$$

where (a_r, a_θ) are the pursuer acceleration components in polar coordinates and the subscript p stands for pursuer P. The angle error is denoted by

$$\varepsilon = \theta_p - \theta_T \tag{2}$$

where the subscript T stands for target T. The pursuer distance from the tracker-to-target line-of-sight is approximated by

$$h \cong r_p(\theta_p - \theta_T) \tag{3}$$

Equation (1b) can be rewritten in terms of h, that is,

$$\ddot h - \frac{\ddot r_m}{r_m}h + (r_p\ddot\theta_T + 2\dot r_p\dot\theta_T) = a_{p_\theta} \tag{4}$$

In our linearized model, the pursuer distance from tracker O is given as a function of time. Using this assumption, the state-space equations are simplified as follows:

$$\begin{cases}\dot h = \nu \\ \dot\nu = \dfrac{\ddot r_m}{r_m}h - (r_p\ddot\theta_T + 2\dot r_p\dot\theta_T) + a_{p_\theta}\end{cases} \tag{5}$$

Optimal Guidance Problem and Solution

Here, the guidance problem is to minimize the following performance index

$$J = \frac{1}{2}\int_{t_0}^{t_f}[b(t)h^2 + R(t)u^2]dt\ ,\quad u = a_{p_\theta} \tag{6}$$

subject to state equation (5) where $b(t)$ and $R(t)$ are positive weighting coefficients and t_f is the predetermined final time. The initial conditions are

$$h(t_0) = h_0\ ,\ \nu(t_0) = \nu_0 \tag{7}$$

Here, four different final conditions are considered as follows:

Case a) $h(t_f) = h^*(t_f),\ \nu(t_f) = \text{free}$

Case b) $h(t_f) = \text{free},\quad \nu(t_f) = \text{free}$

Case c) $h(t_f) = h^*(t_f),\ \nu(t_f) = \nu^*(t_f)$

Case d) $h(t_f) = \text{free},\quad \nu(t_f) = \nu^*(t_f)$

where superscript star denotes the desired final value.

The Hamiltonian is given by

$$H = \frac{1}{2}R(t)u^2 + \frac{1}{2}b(t)h^2 + \lambda_h\nu + \lambda_v\left[\frac{\ddot r_m}{r_m}h + f(t) + u\right] \tag{8}$$

where λ_h and λ_ν are costates and

$$f(t) = -(r_p\ddot\theta_T + 2\dot r_p\dot\theta_T) \tag{9}$$

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The optimality conditions give

$$u = -\text{ª}_{\neg} / R(t) \tag{10}$$

$$\dot{\text{ª}}_h = -b(t)h - \text{ª}_v \frac{\ddot{r}_m}{r_m} \tag{11}$$

$$\dot{\text{ª}}_{\neg} = -\text{ª}_h \tag{12}$$

Therefore, the state equations for states and costates are

$$\begin{cases} \dot{h} = \neg \\ \dot{\neg} = \dfrac{\ddot{r}_m}{r_m} h - (r_p \ddot{\ }_T + 2\dot{r}_p \dot{\ }_T) - \dfrac{\text{ª}_v}{R(t)} \\ \dot{\text{ª}}_h = -b(t)h - \text{ª}_v \dfrac{\ddot{r}_m}{r_m} \\ \dot{\text{ª}}_{\neg} = -\text{ª}_h \end{cases} \tag{13}$$

The above system is linear time-varying, that is,

$$\dot{X} = A(t)X + F(t) \tag{14}$$

$$X(t) = \Phi(t,t_0)X(t_0) + \int_{t_0}^{t} \Phi(t,\text{ª})F(\text{ª})d\text{ª} \tag{15}$$

$$X(t_f) = \Phi(t_f,t)X(t) + \int_{t}^{t_f} \Phi(t,\text{ª})F(\text{ª})d\text{ª} \tag{16}$$

where $A(t)$ is the system matrix, $\Phi(t,t_0)$ is the state transition matrix, and

$$X = [h \ \ \neg \ \ \text{ª}_h \ \ \text{ª}_{\neg}]^T \tag{17}$$

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \ddot{r}_m / r_m & 0 & 0 & -1/R(t) \\ -b(t) & 0 & 0 & -\ddot{r}_m / r_m \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{18}$$

$$F(t) = [0 \ \ f(t) \ \ 0 \ \ 0]^T \tag{19}$$

Therefore, the solution of state equation (13) is

$$h(t) = \Phi_{11}(t,t_0)h_0 + \Phi_{12}(t,t_0)\neg_0 + \Phi_{13}(t,t_0)\text{ª}_{h_0} + \Phi_{14}(t,t_0)\text{ª}_{v_0} + \int_{t_0}^{t} \Phi_{12}(t,\text{ª})f(\text{ª})d\text{ª} \tag{20}$$

$$\neg(t) = \Phi_{21}(t,t_0)h_0 + \Phi_{22}(t,t_0)\neg_0 + \Phi_{23}(t,t_0)\text{ª}_{h_0} + \Phi_{24}(t,t_0)\text{ª}_{v_0} + \int_{t_0}^{t} \Phi_{22}(t,\text{ª})f(\text{ª})d\text{ª} \tag{21}$$

$$\text{ª}_h(t) = \Phi_{31}(t,t_0)h_0 + \Phi_{32}(t,t_0)\neg_0 + \Phi_{33}(t,t_0)\text{ª}_{h_0} + \Phi_{34}(t,t_0)\text{ª}_{v_0} + \int_{t_0}^{t} \Phi_{32}(t,\text{ª})f(\text{ª})d\text{ª} \tag{22}$$

$$\text{ª}_v(t) = \Phi_{41}(t,t_0)h_0 + \Phi_{42}(t,t_0)\neg_0 + \Phi_{43}(t,t_0)\text{ª}_{h_0} + \Phi_{44}(t,t_0)\text{ª}_{v_0} + \int_{t_0}^{t} \Phi_{42}(t,\text{ª})f(\text{ª})d\text{ª} \tag{23}$$

or

$$h(t_f) = \Phi_{11}(t_f,t)h + \Phi_{12}(t_f,t)\neg + \Phi_{13}(t_f,t)\text{ª}_h + \Phi_{14}(t_f,t)\text{ª}_v + \int_{t}^{t_f} \Phi_{12}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{24}$$

$$\neg(t_f) = \Phi_{21}(t_f,t)h + \Phi_{22}(t_f,t)\neg + \Phi_{23}(t_f,t)\text{ª}_h + \Phi_{24}(t_f,t)\text{ª}_v + \int_{t}^{t_f} \Phi_{22}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{25}$$

$$\text{ª}_h(t_f) = \Phi_{31}(t_f,t)h + \Phi_{32}(t_f,t)\neg + \Phi_{33}(t_f,t)\text{ª}_h + \Phi_{34}(t_f,t)\text{ª}_v + \int_{t}^{t_f} \Phi_{32}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{26}$$

$$\text{ª}_v(t_f) = \Phi_{41}(t_f,t)h + \Phi_{42}(t_f,t)\neg + \Phi_{43}(t_f,t)\text{ª}_h + \Phi_{44}(t_f,t)\text{ª}_v + \int_{t}^{t_f} \Phi_{42}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{27}$$

In a compact form, we have

$$X_i(t) = \sum_{j=1}^{4} \Phi_{ij}(t,t_0)X_j(t_0) + \int_{t_0}^{t} \Phi_{i2}(t,\text{ª})f(\text{ª})d\text{ª} \tag{28}$$

or

$$X_i(t_f) = \sum_{j=1}^{4} \Phi_{ij}(t_f,t)X_j(t) + \int_{t}^{t_f} \Phi_{i2}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{29}$$

where X_i is the ith element of the state vector X and Φ_{ij} is the ijth elememt of the state transition matrix. Depending on final conditions, two equations are selected from the above four equations, e.g., for case b, we have $h(t_f) = \text{free}$, $\neg(t_f) = \text{free}$. Since the final time is fixed, we have $\text{ª}_h(t_f) = 0$, $\text{ª}_{\neg}(t_f) = 0$. Therefore, we have a set of two equations with two unknown ª_h and $\text{ª}_{\neg}$, that is,

$$\Phi_{33}(t_f,t)\text{ª}_h + \Phi_{34}(t_f,t)\text{ª}_v = -\Phi_{31}(t_f,t)h - \Phi_{32}(t_f,t)\neg - \int_{t}^{t_f} \Phi_{32}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{30}$$

$$\Phi_{43}(t_f,t)\text{ª}_h + \Phi_{44}(t_f,t)\text{ª}_v = -\Phi_{41}(t_f,t)h - \Phi_{42}(t_f,t)\neg - \int_{t}^{t_f} \Phi_{42}(t_f,\text{ª})f(\text{ª})d\text{ª} \tag{31}$$

Then, $\text{ª}_{\neg}$ is obtained from the above set of two equations with two unknown costates, that is,

$$\begin{bmatrix} \text{ª}_h \\ \text{ª}_v \end{bmatrix} = -\begin{bmatrix} \Phi_{33}(t_f,t) & \Phi_{34}(t_f,t) \\ \Phi_{43}(t_f,t) & \Phi_{44}(t_f,t) \end{bmatrix} \times \begin{bmatrix} \Phi_{31}(t_f,t)h + \Phi_{32}(t_f,t)\neg + \int_{t}^{t_f} \Phi_{32}(t_f,\text{ª})f(\text{ª})d\text{ª} \\ \Phi_{41}(t_f,t)h + \Phi_{42}(t_f,t)\neg + \int_{t}^{t_f} \Phi_{42}(t_f,\text{ª})f(\text{ª})d\text{ª} \end{bmatrix} \tag{32}$$

Finally, the optimal control is obtained from $u = -\text{ª}_{\neg} / R(t)$.

A similar approach is used for the other cases. Hence, the optimal solutions for the mentioned cases are summarized as follows:

Case a) $h(t_f) = h^*(t_f), \neg(t_f) = \text{free}$ [i=1, j=4]:

$$u = c_h h + c_{\neg}\neg + u_d + c_{h_f} h_f^* \tag{33a}$$

Case b) $h(t_f) = \text{free}, \neg(t_f) = \text{free}$ [i=3, j=4]:

$$u = c_h h + c_{\neg}\neg + u_d \tag{33b}$$

Case c) $h(t_f) = h^*(t_f), \neg(t_f) = \neg^*(t_f)$ [i=1, j=2]:

$$u = c_h h + c_{\neg}\neg + u_d + c_{h_f} h_f^* + c_{\neg_f}\neg_f^* \tag{33c}$$

Case d) $h(t_f) = \text{free}, \neg(t_f) = \neg^*(t_f)$ [i=3, j=2]:

$$u = c_h h + c_{\neg}\neg + u_d + c_{\neg_f}\neg_f^* \tag{33d}$$

where the guidance coefficients are

$$c_h = [\Phi_{i3}(t_f,t)\Phi_{j1}(t_f,t) - \Phi_{j3}(t_f,t)\Phi_{i1}(t_f,t)]/D \tag{34}$$

$$c_{\neg} = [\Phi_{i3}(t_f,t)\Phi_{j2}(t_f,t) - \Phi_{j3}(t_f,t)\Phi_{i2}(t_f,t)]/D \tag{35}$$

$$c_{h_f} = \Phi_{j3}(t_f,t)/D \tag{36}$$

$$c_{\neg_f} = -\Phi_{i3}(t_f,t)/D \tag{37}$$

$$D = [\Phi_{i3}(t_f,t)\Phi_{j4}(t_f,t) - \Phi_{j3}(t_f,t)\Phi_{i4}(t_f,t)]R(t) \tag{38}$$

Also, the control term related to moving beam is given by

$$u_d = \Phi_{i3}(t_f,t)\int_t^{t_f}\Phi_{j2}(t_f,\tau)f(\tau)d\tau/D - \Phi_{j3}(t_f,t)\int_t^{t_f}\Phi_{i2}(t_f,\tau)f(\tau)d\tau/D \quad (39)$$

For time-invariant system, we have $\Phi_{ij}(t_f,t)=\Phi_{ij}(t_{go})$ where t_{go} is the time-to-go the final time $(t_{go}=t_f-t)$. The time-to-go until the final time is approximated by the pursuer-target relative range divided by closing velocity.

Cases a and b can be utilized for LOS guidance, but Cases c and d with final normal velocity may be preferred for trajectory tracking. The final trajectory angle can be applied by the final normal velocity. Also, Cases c and d may be used for LOS guidance with final dive angle. Moreover, as a modified LOS guidance against moving target, the final normal velocity can be computed from collision triangle.

Special case

Consider a special case in which the target angular velocity is constant. Also, the pursuer range rate is assumed constant, that is,

$$\dot{r}_p = V_p\cos\theta_a \quad (40)$$

where V_p is the pursuer speed, θ is the angle between the pursuer velocity and the tracker-pursuer LOS, and θ_a is the average value of θ between current time and the final time. To calculate u_d, we assumed θ_a to be constant. Therefore,

$$f(t) = -2V_p\omega_T\cos\theta_a \quad (41)$$

Using the following property of the state transition matrix:

$$\frac{d}{dt}\Phi(t_f,t) = -\Phi(t_f,t)A(t) \quad (42)$$

and assuming $R(t)$ to be constant, we can obtain

$$\int_t^{t_f}\Phi_{i2}(t_f,\tau)d\tau = R[\Phi_{i4}(t_f,t_f)-\Phi_{i4}(t_f,t)] \quad (43)$$

Also,

$$\Phi_{i4}(t_f,t_f) = \begin{cases}1 & \text{for } i=4\\ 0 & \text{for } i\neq 4\end{cases} \quad (44)$$

Therefore,

$$\int_t^{t_f}\Phi_{k2}(t_f,\tau)f(\tau)d\tau = -2RV_p\omega_T\cos\theta_a \times \begin{cases}-\Phi_{k4}(t_f,t) & \text{for } k\neq 4\\ 1-\Phi_{44}(t_f,t) & \text{for } k=4\end{cases} \quad (45)$$

For Cases c and d, we have $i,j\neq 4$. In this case the relation for u_d simplifies to

$$u_d = 2V_p\omega_T\cos\theta_a \quad (46)$$

For Cases a and b, we obtain

$$u_d = 2V_p\left[1-\frac{\Phi_{i3}(t_f,t)}{D}\right]\omega_T\cos\theta_a \quad (47)$$

Concluding Remarks

A class of closed-loop optimal line-of-sight guidance laws is developed for moving targets assuming a point mass pursuer having perfect control system. The solution is obtained using linearized model in which the pursuer range is given as a given function of time and the maneuvering acceleration is applied normal to LOS. Since, the governing equations are uncoupled in the linearized model, the solution can be extended to two-dimensional case. Moreover, different final conditions are considered in the solution, i.e., final position and final normal velocity are applied fixed or free, depending on application. Here, the final angle constraint is applied by final normal velocity for better trajectory tracking problem.

Appendix A

The state transition matrix can be solved for special cases. The system matrix is simplified by assuming $\ddot{r}_m = 0$, that is,

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1/R(t)\\ -b(t) & 0 & 0 & 0\\ 0 & 0 & -1 & 0\end{bmatrix} \quad (48)$$

Also, the state transition matrix for $R(t)=1$ and b as a positive constant is simply obtained using the following relation:

$$\Phi(t) = L^{-1}\left[(sI-A)^{-1}\right] \quad (49)$$

where s is the Laplace domain variable and L^{-1} denotes the Laplace inverse transform.

$$\Phi_{11}(t) = \cosh\mu t\,\cos\mu t \quad (50)$$

$$\sqrt{2}b^{1/4}\Phi_{12}(t) = \cosh\mu t\,\sin\mu t + \sinh\mu t\,\cos\mu t \quad (51)$$

$$\sqrt{2}b^{3/4}\Phi_{13}(t) = \cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (52)$$

$$\sqrt{b}\,\Phi_{14}(t) = \sinh\mu t\,\sin\mu t \quad (53)$$

$$\sqrt{2}b^{-1/4}\Phi_{21}(t) = -\cosh\mu t\,\sin\mu t + \sinh\mu t\,\cos\mu t \quad (54)$$

$$\Phi_{22}(t) = \cosh\mu t\,\cos\mu t \quad (55)$$

$$\sqrt{b}\,\Phi_{23}(t) = \sinh\mu t\,\sin\mu t \quad (56)$$

$$\sqrt{2}b^{1/4}\Phi_{24}(t) = -\cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (57)$$

$$\sqrt{2}b^{-3/4}\Phi_{31}(t) = -\cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (58)$$

$$\Phi_{32}(t) = -\sqrt{b}\sinh\mu t\,\sin\mu t \quad (59)$$

$$\Phi_{33}(t) = \cosh\mu t\,\cos\mu t \quad (60)$$

$$\sqrt{2}b^{-1/4}\Phi_{34}(t) = \cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (61)$$

$$\Phi_{41}(t) = \sqrt{b}\sinh\mu t\,\sin\mu t \quad (62)$$

$$\sqrt{2}b^{-1/4}\Phi_{42}(t) = \cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (63)$$

$$\sqrt{2}b^{1/4}\Phi_{43}(t) = -\cosh\mu t\,\sin\mu t - \sinh\mu t\,\cos\mu t \quad (64)$$

$$\Phi_{44}(t) = \cosh\mu t\,\cos\mu t \quad (65)$$

where

$$\mu = b^{1/4}/\sqrt{2} \quad (66)$$

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